Transient Heat Transfer Calculations

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Energy In = Energy Out

Energy In = Heat Stored + Heat Conducted

Solar Gain= Heat Stored + Heat Conducted

$$\frac{dS}{dt} = C\frac{dT}{dt} + K\frac{dT}{dx}$$

Where S is solar heat gain, C is the capacitance (mass multiplied by specific heat), and K is the conductivity.

$\{S\} = \{C\}\{\frac{dT}{dt}\} + [K]\{\frac{dT}{dx}\}$

$\{S\} = \{C\}\{\frac{dT}{dt}\} + [K]\{\frac{dT}{dx}\}$

$\{C\}\{\frac{dT}{dt}\} = \{S\} - [K]\{\frac{dT}{dx}\}$

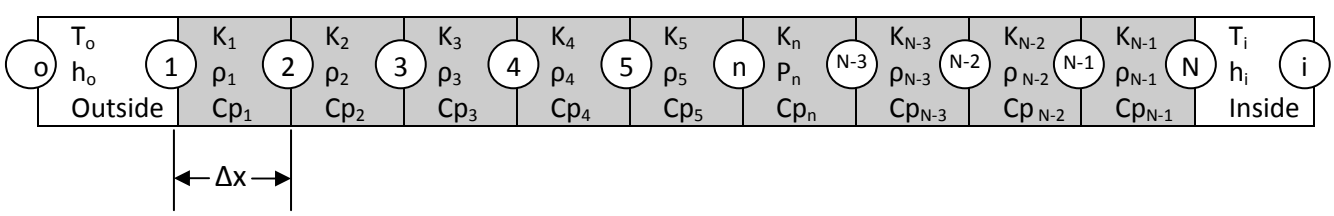
$\{\frac{dT}{dt}\} = \{C\}^{-1}(\{S\} - [K]\{\frac{dT}{dx}\})$

$\{(T_{t+dt} - T_t)/dt\} = \{C\}^{-1}(\{S\} - [K]\{\frac{dT}{dx}\})$

$\{(T_{t+dt} - T_t)\} = \{C\}^{-1}(\{S\} - [K]\{\frac{dT}{dx}\})dt$

$\{T_{t+dt}\} = \{C\}^{-1}(\{S\} - [K]\{\frac{dT}{dx}\})dt + \{T_t\}$

$\{T_{t+dt}\} = \{C\}^{-1}(\{S\} - [K]\{T\}/\Delta x)\,\Delta t + \{T_t\}$



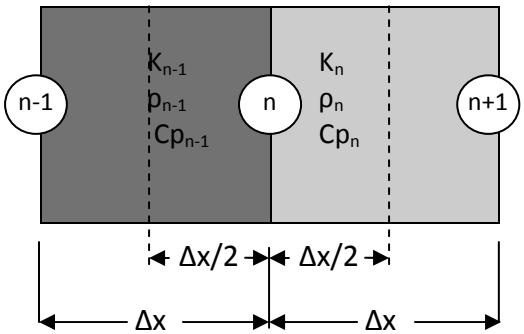
Use the following to determine the heat conductivity matrix [K].

Heat transfer "Outside" caused by rise in outside temperature, all other temperatures equal and unchanged.	$h_o(T_o - T_1)$
Heat Transfer at node 1, caused by rise in temperature at node 1, all other temperatures equal and unchanged.	$h_o(T_o - T_1) + K_1(T_1 - T_2)$
Heat Transfer at node 2, caused by rise in temperature at node 2, all other temperatures equal and unchanged.	$K_1(-T_1 + T_2) + K_2(T_2 - T_3)$
...	$K_2(-T_2 + T_3) + K_3(T_3 - T_4)$
...	$K_3(-T_3 + T_4) + K_4(T_4 - T_5)$
...	$K_4(-T_4 + T_5) + K_5(T_5 - T_6)$
...	$K_{(n-1)}(-T_{(n-1)} + T_{(N-3)}) + K_n(T_{(n-1)} - T_n)$
...	$K_{(N-4)}(-T_{(N-4)} + T_{(N-3)}) + K_{(N-3)}(T_{(N-3)} - T_{(N-2)})$
...	$K_{(N-3)}(-T_{(N-3)} + T_{(N-2)}) + K_{(N-2)}(T_{(N-2)} - T_{(N-1)})$
...	$K_{(N-2)}(-T_{(N-2)} + T_{(N-1)}) + K_{(N-1)}(T_{(N-1)} - T_N)$
...	$K_{(N-1)}(-T_{(N-1)} + T_N) + h_i(T_N - T_i)$
Heat transfer "Inside" caused by rise in inside temperature, all other temperatures equal and unchanged.	$h_i(-T_N + T_i)$

The above formulas can be represented in matrix form [K]{T} as the following:

		o	1	2	3	4	5	...	N-3	N-2	N-1	N	i	
	o	h_o	-ho	0	0	0	0	0	0	0	0	0	0	T_o
	1	-ho	ho + K1	-K1	0	0	0	0	0	0	0	0	0	T_1
	2	0	-K1	K1 + K2	-K2	0	0	0	0	0	0	0	0	T_2
	3	0	0	-K2	K2 + K3	-K3	0	0	0	0	0	0	0	T_3
	4	0	0	0	-K3	K3 + K4	-K4	0	0	0	0	0	0	T_4
	5	0	0	0	0	-K4	K4 + K5	-K5	0	0	0	0	0	T_5
	...	0	0	0	0	0	-K(n-1)	K(n-1) +Kn	-Kn	0	0	0	0	$T_{...}$
	N-3	0	0	0	0	0	0	-K(N-4)	K(N-4) +K(N-3)	-K(N-3)	0	0	0	T_{N-3}
	N-2	0	0	0	0	0	0	0	-K(N-3)	K(N-3) +K(N-2)	-K(N-2)	0	0	T_{N-2}
	N-1	0	0	0	0	0	0	0	0	-K(N-2)	K(N-2) +K(N-1)	-K(N-1)	0	T_{N-1}
	N	0	0	0	0	0	0	0	0	0	-K(N-1)	K(N-1) +hi	-hi	T_N
	i	0	0	0	0	0	0	0	0	0	0	-hi	h_i	T_i

Capacitance matrix {C} is the specific heat multiplied by the mass of each node. The mass of each node is equal to the half the mass of the block on each side of the node.



Mass at Node = Density x Volume = Density x Unit Area x Thickness = Density x Unit Area x dx/2

$$\{C\}\{T\} = \begin{array}{c} o \\ 1 \\ 2 \\ n \\ N\text{-}1 \\ N \\ i \end{array} \begin{bmatrix} \rho_o Cp_o \\ \rho_1 Cp_1/2 \\ \rho_1 Cp_1/2 + \rho_2 Cp_2/2 \\ \rho_{(n-1)} Cp_{(n-1)}/2 + \rho_n Cp_n/2 \\ \rho_{(N-2)} Cp_{(N-2)}/2 + \rho_{(N-1)} Cp_{(N-1)}/2 \\ \rho_{(N-1)} Cp_{(N-1)}/2 \\ \rho_i Cp_i \end{bmatrix} \begin{bmatrix} T_o \\ T_1 \\ T_2 \\ T_3 \\ T_{N-1} \\ T_N \\ T_i \end{bmatrix} \Delta x \qquad \{C\}^{-1} = \begin{array}{c} o \\ 1 \\ 2 \\ n \\ N\text{-}1 \\ N \\ i \end{array} \begin{bmatrix} 1/(\rho_o Cp_o) \\ 2/\rho_1 Cp_1 \\ 2/(\rho_1 Cp_1 + \rho_2 Cp_2) \\ 2/(\rho_{(n-1)} Cp_{(n-1)} + \rho_n Cp_n) \\ 2/(\rho_{(N-2)} Cp_{(N-2)} + \rho_{(N-1)} Cp_{(N-1)}) \\ 2/(\rho_{(N-1)} Cp_{(N-1)}) \\ 1/(\rho_i Cp_i) \end{bmatrix} / dx$$

To find the new T for each new Δt, multiply $\{C\}^{-1}$ times ({S} - [K]{T_t}/Δx) Δt + {T_t}

$\{T_{t+dt}\} = \{C\}^{-1} (\{S\} - [K]\{T\}/\Delta x)\ \Delta t + \{T_t\}$