John von Neumann, the Mathematician

Domokos Szász

Imagine a poll to choose the best-known mathematician of the twentieth century. No doubt the winner would be John von Neumann. Reasons are seen, for instance, in the title of the excellent biography [M] *by Macrae:* John von Neumann. The Scientific Genius who Pioneered the Modern Computer, Game Theory, Nuclear Deterrence, and Much More. *Indeed, he was a fundamental figure not only in designing modern computers but also in defining their place in society and envisioning their potential. His minimax theorem, the first theorem of game theory, and later his equilibrium model of economy, essentially inaugurated the new science of mathematical economics. He played an important role in the development of the atomic bomb. However, behind all these, he was a brilliant mathematician. My goal here is to concentrate on his development and achievements as a mathematician and the evolution of his mathematical interests.*

The Years in Hungary (1903-1921)

In 1802, the Hungarian mathematician János Bolyai was born in Kolozsvár (called Cluj today). János Bolyai was the first Hungarian mathematician, maybe the first Hungarian scholar, of world rank. His non-euclidean hyperbolic geometry, also discovered at the same time independently by the Russian mathematician Lobachevsky, is now everywhere recognized, but could not be appreciated in his time in Hungary. He never had a job as a mathematician. In the early nineteenth century, Hungary was in the hinterlands of world mathematics.

A century later, John von Neumann was born as Neumann János on December 28, 1903, in Budapest. His family belonged to the cultural elite of the city. Not only were both of his parents' families quite well-to-do, they understood how to use their wealth to live a rich and complete human life. But beyond all this, Hungary had changed dramatically during the 101 years since Bolyai's birth. In 1832, János Bolyai's discovery was not understood by anyone in the country, except for his father. In contrast, János Neumann's exceptional talent was discovered very early and was nurtured by top mathematicians on such a level that it is hard to imagine better circumstances for a prodigy in any other part of the world.

Much has been written on the life of Neumann and his achievements. I have used extensively Norman Macrae's exciting biography [M], which I recommend for the general reader. For the mathematician, I recommend the special issue 3 of Volume 64 (1958) of the *Bulletin of the American Mathematical Society* dedicated to von Neumann shortly after his death. In this issue, first of all, Stanisław Ulam, a lifelong friend and collaborator of Johnny, as he was called by his friends, gives a brief biography and concise mathematical overview (see also Ulam's autobiography *Adventures of a Mathematician*). After this introductory paper, world experts in various fields place von Neumann's achievements in a wider context. Excellent papers with a similar aim also appeared in the Hungarian periodical *Matematikai Lapok*, unfortunately in Hungarian only. I also mention the book of W. Aspray *John von Neumann and the origins of Modern Computing* [A]. I note that von Neumann's works were edited by A. H. Taub [T]. A selection was edited by F. Bródy and T. Vámos [BV].

Neumann's father, Neumann Miksa (Max Neumann), a doctor of law, prospered as a lawyer for a bank. "He was a debonair, fourth-generation-or-earlier, non-practicing Hungarian Jew, with a fine education in a Catholic

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provincial high-school. He was well attuned to *fin-de-siècle* Austria-Hungary and indeed to being a lively intellectual in any age. His party piece was to compose two-line ditties about the latest vicissitudes of his personal or business life, and about national and international politics."[1] Jacob Kann, Neumann's grandfather through his mother Kann Margit (Margaret Kann), "was a distinctly prosperous man. With a partner, he had built up in Budapest a thriving business in agricultural equipment that grew even faster in 1880-1914 than Hungary's then soaring GNP." The Kanns owned a rather big and nice house close to the Hungarian Parliament on the so-called Kiskörút, Small Boulevard. The Kanns and later the families of their four daughters lived there, while the ground floor was occupied by the firm Kann-Heller. When the families grew larger, they also rented the neighboring house. János's mother Margit was "a mother hen tending her children and protecting them. She was a family woman, a good deal less rigorous than her husband, artistically inclined, a wafer-thin and later chain-smoking enthusiast for comporting oneself with what she called 'elegance' (which later became Neumann's highest term of praise for neat mathematical calculations such as the ones that made the H-bomb possible)."

Max and Margaret had three children, of which Johnny was the oldest. "Grandfather Jacob Kann had gone straight from commercial high school into founding his business, but he proved demonic in his capacity for arithmetic manipulation. He could add in his head monstrous columns of numbers or multiply mentally two numbers in the thousands or even millions. The six-year-old Johnny would laboriously perform the computations with pencil and paper, and announce with glee that Grandfather had been absolutely on the mark. Later Johnny himself was known for his facility in mental computation, but he had long before persuaded himself that he could never match Jacob's level of multiplication skill."

"Nursemaids, governesses and preschool teachers were an integral part of upper-middle-class European households in those days, especially in countries where (as in Hungary) children in such families did not start school until age ten. As the Kann grandchildren and some of Johnny's second cousins came along, the building on the Small Boulevard became an educational institution in its own right. There was an especially early emphasis on learning foreign languages. Father Max thought that youngsters who spoke only Hungarian would not merely fail in the Central Europe then darkening around them, they might not even survive." So Johnny (or *Jancsi* in Hungarian) learned German, French, English, and Italian from various governesses from abroad. Johnny later told colleagues in Princeton that as a six-year-old he would converse with his father in classical Greek. This seems to have been a joke, but his knowledge of languages was real. It not only made his communication easier in Germany, Zürich, and in the US, but it may have imprinted at an early age axiomatic and abstract, algorithmic thinking into his brain.

In such an environment, young Johnny was easily drinking in the knowledge surrounding him. One of his special interests was history: The family bought "an entire library, whose centerpiece was the *Allgemeine Geschichte* by Wilhelm Oncken. Johnny ploughed through all forty-four volumes. Brother Michael was confounded by the fact that what Johnny read, Johnny remembered. Decades later friends were startled to discover that he remembered still."

Between 1914 and 1921, Johnny attended the Lutheran Gymnasium in Budapest. Here, too, he received a superb education. This was made possible by the fast economic and social progress after the compromise with Austria[2] and subsequent cultural and educational reforms.[3]

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[1]Here and below, many unattributed quotes are from [M].

[2]**Some data from the late nineteenth-century history of Hungary**. *1867*: Appeasement with Austria and the formation of the Austro-Hungarian Monarchy after the defeat of the 1848 revolution. *1873*: Formation of the city Budapest from smaller cities such as Buda and Pest. In the years afterward, Hungary and Budapest experienced extremely fast economic, social, and intellectual progress. Society was quite open, with "a flood of Jewish immigration into Hungary in the 1880s and 1890s as there was simultaneously to New York." In 1896 a World Exposition was held in Budapest also commemorating the arrival of Hungarians to the Carpathians in 896. Hausmann-like reconstruction of the city contributed to its present character. For instance, the first subway of the European continent was inaugurated in 1896 in Budapest. "In 1903 the Elisabeth Bridge over the Danube was the longest single span bridge in the world." For the interested reader I strongly recommend the book of John Lukacs [Luk].

[3]**Some data from the nineteenth-century cultural progress of Hungary**. *End of nineteenth and first half of twentieth century*: Development of national thought and nationalistic institutions, in particular, the foundation of the National Museum and the Hungarian Academy of Sciences. After the Appeasement of 1867, the educational system was transformed and modernized. (For some time the Minister of Education was Loránd Eötvös, the famous physicist, inventor of the torsion pendulum). The *Gazette of Sciences*, called today *World of Nature*, was founded in 1869, that is, in the same year as the magazine *Nature* in Britain! The Mathematical and Physical Society was founded in 1891. (Since 1947 there are separate societies for Mathematics and Physics, called the János Bolyai and the Loránd Eötvös Societies, respectively; in 1968 the John von Neumann Computer Society was founded.) In 1894 Dániel Arany founded the *Középiskolai Matematikai Lapok*, the oldest journal for high-school students; this journal still flourishes in an extended form including physics. In the same year 1894, the first Mathematical and Physical Competitions started (today the mathematical competitions are called József Kürschák Competitions).

As a result of these reforms the whole educational system was strong; for the talented (and let us also add, wealthy) pupils, there were élite high-schools. One of them was the Lutheran Gymnasium. It was quite liberal and had a number of Jewish pupils. Nobel Prize winners E. Wigner and later J. Harsányi also attended this high-school. As a matter of fact, Wigner studied a year above Neumann, and already at school a respect and a friendship were formed between the two.

Johnny's teacher of mathematics was the celebrated László Rátz. Between 1896 and 1914, he was the chief editor of the high-school journal *Középiskolai Matematíkaí Lapok*. "Wigner and others recall that Rátz's recognition of Johnny's mathematical talent was instant." Rátz agreed with Max Neumann that Johnny would be supervised by professional mathematicians from the universities. Thus Professor J. Kurschak (Kürschák József) from Technical University arranged that Johnny would be tutored by the young G. Szego (Szegő Gábor) in 1915–1916. Later he was also taught by M. Fekete (Fekete Mihály) and Leopold Fejér (Fejér Lipót), and could also talk to A. Haar (Haar Alfréd) and Frédéric Riesz (Riesz Frigyes). All in all, the prodigy received education and supervision on the highest professional and intellectual level.

In the years 1919–1921, when von Neumann graduated from the Lutheran Gymnasium, the mathematical competition for secondary schools did not take place because of the revolution in Hungary; but in 1918, Neumann was permitted to sit in as an unofficial participant, and would have won the first prize. The list of winners of the mathematical competitions before 1928 include among others L. Fejér, Th. von Kármán, D. Konig, A. Haar, M. Riesz, G. Szego, and E. Teller, whereas L. Szilárd won a second prize.

It is commonplace that around and shortly after the turn of the nineteenth and twentieth centuries Hungary exported a tremendous amount of brains to the West besides von Neumann, but it may be worth listing some of these scientists: Dennis Gabor (Gábor Dénes; Nobel Prize 1971), John Harsányi (Harsányi János; Nobel Prize 1994), Georg von Hevesy (Hevesy György; Nobel Prize 1943), Theodore von Kármán (Kármán Tódor), Nicholas Kurti (Kürti Miklós), Cornelius Lanczos (Lánczos Kornél), Peter Lax (Lax Péter; Wolf Prize 1987), George Olah (Oláh György; Nobel Prize 1994), Michael Polanyi (Polányi Mihály), George Pólya (Pólya György), Gabor Szego (Szegő Gábor), Albert Szent-Györgyi (Szent-Györgyi Albert; Nobel Prize 1937), Leo Szilárd (Szilárd Leó), Edward Teller (Teller Ede), Eugene Wigner (Wigner Jenö; Nobel Prize 1963), and Aurel Friedrich Wintner. One might wonder about the "von"s figuring in several of these names. Macrae explains the case of Neumann: "In 1913 the forty-three-year-old Max was rewarded for his services to the government: he received the noble placename Margittai with hereditary nobility, so that in German he and his descendants could be called 'von Neumann.' Ennoblement was not an unusual award for prominent bankers and industrialists during those last years of the Austro-Hungarian Empire. Many of the 220 Hungarian Jewish families who were ennobled in 1900–1914 (vs. just over half of that number in the whole century before) hastened to change their names. Ennoblement was a way through which one could seize the chance to call himself less Jewish. Max Neumann deliberately did not change his name."

The house on Kiskörút today. (Photo by Domokos Szász)

The Mathematics of John von Neumann

My aim here should certainly be quite modest, and I can go into a detailed description neither of his achievements nor of their influence. My goal is to follow some of the main and continuing interests of John von Neumann. The selection of these interests is subjective and reflects my knowledge and judgement.

Despite the breadth of his interests, the picture is not as complicated as it looks at first glance. This is especially true for the years before the War. It seems he usually had one main interest, and thought about other problems with the left hand, so to speak. This does not mean that the other results are less important, only that even a genius is subject to laws of nature. If one wants to reach repeated breakthroughs in a problem or more than one (cf. Poincaré), then a necessary condition is full concentration on it for quite a period. In short, the genius is known both by the difficulty of the problems he can solve with full concentration, and by the difficulty of the problems he can solve with perhaps less persistent concentration.

Axiomatic Set Theory

In the first part of this paper I emphasized that this exceptional prodigy received an optimal launch in Hungary. In addition, one should note that the excellent mathematical support in Budapest ensured his start as a scholar and as a researcher. Apart from his first paper with Fekete, his big interest was the axiomatization of set theory. It is most likely that he heard about this problem from a tutor in Budapest. Julius König (König Gyula) (the father of the graph-theoretician Denes König (König Dénes)) was himself also working on set theory, and in particular on the continuum hypothesis, but he died in 1913, before Neumann entered high-school. Anyway, mathematicians in Budapest were definitely aware of this circle of problems (J. Kurschak, Neumann's mentor, for instance, was a colleague of J. König). Macrae writes: "The second of Johnny's papers (about transfinite ordinals) had been prepared while he was still in high-school in 1921, although it was not published until 1923"(see [vN23]).

The young Neumann János. (Photo from http://www.ysfine.com.wigner/neum/ymath.jpeg)

His papers on set theory and logic were published in 1923 (1), 1925 (1), 1927 (1), 1928 (2), 1929 (1), and 1931 (2). It is known that when he read about Gödel's undecidability theorem in 1931, he dropped thinking seriously about this field. But there was another reason for that, too: starting at the latest from his scholarship in Göttingen in 1926, his main interest had changed: it became centered around laying down the mathematical foundations of quantum mechanics and hence around functional analysis.

Mathematical Foundations of Quantum Physics

Hilbert certainly heard about the Hungarian Wunderkind quite early, and highly respected his results on set theory and on Hilbert's proof theory. But in the mid-twenties, and very much in Göttingen, quantum mechanics was in the center of attention. Hilbert himself was strongly attracted to this disputed science. One should not forget that Göttingen was a center of quantum mechanics: Max Born (Nobel Prize 1954) and J. Franck (Nobel Prize 1925) were professors there between 1921 and 1933 and between 1920 and 1933, respectively, and W. Heisenberg (Nobel Prize 1932) and W. Pauli (Nobel Prize 1945) also spent some years there. In 1927, a long and important paper [HNvN27] on foundations of quantum mechanics of Neumann with Hilbert and L. Nordheim appeared. It also formulated a program that was later carried out in von Neumann's book [vN32b].

The program became a success story in many ways. It meant a victory for the Hilbert-space approach. Moreover, it attracted the attention of mathematicians to the theory of operators and to functional analysis. Also, by translating the problems of quantum physics to the language of mathematics, it formulated intriguing questions, which either arose in physics or were generated by them by the usual process of mathematics. A substantial result of von Neumann was the spectral theory of unbounded operators, generalizing that given by Hilbert for bounded operators. Also, the language of operator theory helped to reconcile the complementary and—apparently contradictory—approaches of Heisenberg and of Schrödinger.

Von Neumann's papers on the mathematical foundations of quantum mechanics and functional analysis appeared in 1927 (4), 1928 (5), 1929 (6), 1931 (2), 1932 (2), 1934 (2), 1935 (3), and 1936 (1). I add that physicists appreciate, in particular, his theories of hidden variables (more exactly, his proof of their nonexistence), of quantum logic, and of the measuring process (cf. Geszti's article in [BV]). From the physical and also from gnoseological point of view I stress his disproof of the existence of hidden variables. The laws of quantum physics are by their very nature stochastic, basically contrary to the deterministic Laplacian view of the universe. Several scholars, including Albert Einstein (his saying "God does not throw dice" became famous), did not accept a nondeterministic universe. There was a belief that probabilistic laws are always superficial and that behind them there must be some hidden variables, by the use of which the world becomes deterministic. This belief was irreversibly repudiated by von Neumann.[4]

Von Neumann writes in 1947 in an article entitled *The Mathematician* [vN47], "It is undeniable that some of the best inspirations of mathematics—in those parts of it which are as pure mathematics as one can imagine—have come from the natural sciences." It is certainly undeniable that having shown "his lion's claws" in set theory, Neumann got to the right place: To elaborate the mathematical foundations of quantum mechanics was the best imaginable challenge for the young genius. The completion of this task led to a victorious period for functional analysis, the theory of unbounded operators. Von Neumann kept up this interest until his death. Peter Lax remembers, "I recall how pleased and excited Neumann was in 1953 when he learned of Kato's proof of the self-adjointness of the Schrödinger operator for the helium atom." [Lax].

Von Neumann, being a true mathematician, was also aware of the other fundamental motivation of a mathematician. In the same 1947 article [27] he starts, "I think it is a relatively good approximation to truth ... that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure." Then he continues, "But, once they are so conceived, the subject begins to live a peculiar life of his own and is better compared to a creative one, governed by almost entirely aesthetical motivations, than to anything else and, in particular, to an empirical science." I can not resist adding one more idea of von Neumann about his favourite criterion of *elegance*: "One expects from a mathematical theorem or from a mathematical theory not only to describe and to classify in a simple and elegant way numerous and a priori special cases. One also expects 'elegance' in its 'architectural' structural makeup. Ease in stating the problem, great difficulty in getting hold of it and in all attempts at approaching it, then again some very surprising twist by which the approach becomes easy, etc. Also, if the deductions are lengthy or

[4] I learnt from Peter Lax that von Neumann made a subtle error about hidden variables. It was fixed up by S. Kochen and E. P. Specker in *J. of Math.& Mech*. 17 (1967), 59-87. Editor's note: This issue remains contentious, and will be revisited in a future issue of *The Mathematical Intelligencer*.

complicated, there should be some simple general principle involved, which 'explains' the complications and detours, reduces the apparent arbitrariness to a few simple guiding motivations, etc."

von Neumann Algebras

In von Neumann's hands, operator theory started to live an independent life in 1929. In that year he published a long paper [vN29] in *Math. Ann.*, whose first half was his initial work on algebras of operators. This was a completely new branch of mathematics, whose study he continued in a couple of papers published in 1936 (3), 1937 (1), 1940 (1), and in 1943 (2) (these include his classic four-part series of works with F. J. Murray). Most likely von Neumann had no external motivation, for instance from physics, at least he does not say so in his first paper. He, partially with Murray, obtained his celebrated classification, leaving the classification of type III algebras open (this is what Alain Connes completed later). The creation of the theory of von Neumann algebras shows that his unrivalled knowledge, speed of thinking, and intuition led him to a brilliant discovery, whose real value only became evident three decades after its invention. One important idea was that the dimension of an algebra (or of a space) is strongly related to the invariance group acting on the object. The theory of von Neumann algebras started to flourish around the 1960s with many deep results, and, in particular, with the Tomita-Takesaki theory also revealing profound links of von Neumann algebras to physics. Not much later their relation to field theories was also discovered. Subsequently, this theory became deeper, and absolutely new connections and applicability were also discovered. In fact, at least four Fields Medals have been awarded for radically new results on or in connection with von Neumann algebras: to A. Connes in 1982, to V. F. R. Jones and to E. Witten in 1990, and to M. Kontsevich in 1998. It is intriguing that Jones's discovery of the famous Jones polynomials, a topological concept related to knots/braids, was motivated by ideas from von Neumann algebras. The easier and more natural topological construction was afterward guessed by Witten and rigorously executed by Kontsevich. See [AI] for lectures of the first three medalists and by H. Araki, J. S. Birman, and L. Fadeev for their laudations. See also http://math.berkeley.edu/~vfr/.

Despite the fact that the revolutionary progress of the theory of von Neumann algebras only occurred in the 1960s, von Neumann was conscious of their potential importance. In 1954, answering a questionnaire of the National Academy of Sciences, he selected as his most important scientific contributions 1) his work on mathematical foundations of quantum theory, 2) his theory of operator algebras, and 3) his work in ergodic theory.

"Side" Interests, Ergodic Theory, and Game Theory Among Others

Ergodic Theory. The appearance of ergodic theory in the aforementioned list is somewhat surprising. To be sure, von Neumann was well aware of the importance of the ergodic theorem for the foundations of statistical mechanics. (The ergodic hypothesis grew from ideas of Maxwell, Boltzmann, Kelvin, Poincaré, Ehrenfest, etc.) As a matter of fact, von Neumann found and proved the first-ever ergodic theorem: the L_2 version in 1931 (which only appeared in 1932) [vN32a]. Then George Birkhoff established the individual ergodic theorem in 1932. Yet despite his numerous papers on ergodic theory (in 1927 (1), 1932 (3), 1941 (1), 1942 (1), and 1945 (1)), it seems to me that it was not one of his most persistent interests. Furthermore, he may have felt it as less than a triumph. When he heard about Birkhoff's result "Johnny expressed pleasure rather than resentment, although he did kick himself for not spotting the next steps from his calculations that Birkhoff saw."

Progress in applying the notion of ergodicity to statistical mechanics, that is, showing that interesting mechanical systems are ergodic, has been rather slow. Only in 1970 could Sinai [S] find the first true mechanical system, that of two hard discs on the 2-torus, whose ergodicity he could prove; he conjectured that the situation is similar for any number of balls in arbitrary dimension. In 1999 Simányi and I showed that typical hard-ball systems of N balls of masses $m_1, \ldots, m_N$ and radii r moving on the v-torus are hyperbolic; in 2006 Simányi established ergodicity in general under an additional hypothesis: the Chernov-Sinai Ansatz.

Other Interests. Von Neumann's intelligence, culture, speed, motivation, and communication abilities were at play in many fields from the very start. He touched upon algebra, theory of functions of real variables, measure theory, topology, continuous groups, lattice theory, continuous geometry, almost periodic functions, representation of groups, quantum logic, etc. I suspect in a sense these were done "with his left hand." Nevertheless, for his two-part memoir ([vN34] and [BvN35]) on almost periodic functions and groups, he received the prestigious Bôcher Prize in 1938. He would have liked to construct invariant measures for groups, a goal actually completed by A. Haar in 1932. Von Neumann returned several times to this question, and the idea of the construction was also exploited in the construction of the dimension in his theory of operator algebras.

Game Theory and Mathematical Economics. Two related left-hand topics deserve special mention: game theory and mathematical economics. It is well known that von Neumann's two game-theory papers in 1928 (the more detailed one is [vN28]) contained the formulation and the proof of the minimax theorem (a model of symmetric, two-person games had already been suggested by Émile Borel in 1921, but he had doubts about the validity of a minimax theorem). Also, in 1937 von Neumann published a model of general economic equilibrium [vN37], known today by his name. These two sources were basic for his big joint enterprise with O. Morgenstern, the fundamental monograph [vNM44] *Theory of Games and Economic Behavior*. These topics were also the objects of his papers in 1950 (1), 1953 (3), 1954 (1), and 1956 (1) (and two further unfinished manuscripts that appeared in 1963). Judging from the fact that in 1994 three Nobel laureates in economics were named for achievements in game theory (J. Harsányi, J. Nash, and Reinhart Selten), it is not too bold to surmise that had Neumann lived until this Nobel Prize was founded in 1969, he would have been the first to receive it.

Von Neumann's Mean Ergodic Theorem

Domokos Szász

Ludwig Boltzmann's ergodic hypothesis, formulated in the 1870s, says: *for large systems of interacting particles in equilibrium, time averages of observables are equal to their ensemble averages (i.e., averages w.r.t. the time-invariant measure)* (cf. D. Szász, Boltzmann's Ergodic Hypothesis, a Conjecture for Centuries? *Hard Ball Systems and the Lorentz Gas*, Springer Encyclopaedia of Mathematical Sciences, vol. 101, 2000, pp. 421-446.). For Hamiltonian systems the natural invariant measure is the Liouvillian one. In 1931, Koopman (B. O. Koopman, Hamiltonian systems and transformations in Hilbert space, *Proc. Nat. Acad. Sci.* vol. 17 (1931) pp. 315-318.) made a fundamental observation: If T is a transformation of a measurable space X keeping the measure μ invariant, then the operator U defined by $(Uf)(x) := f(Tx)$ is unitary on L_2. This functional-analytic translation of the ergodic problem led von Neumann in the same year to finding his famous

MEAN ERGODIC THEOREM

Theorem 1 (John von Neumann, Proof of the quasi-ergodic hypothesis, *Proc. Nat. Acad. Sci.* vol. 18 (1932) pp. 70-82.)

For $f \in L_2(\mu)$ the L_2-limit

$$\lim_{n\to\infty} \frac{1}{n}\left(f + f(Tx) + \cdots + f(T^{n-1}x)\right) = F(x) \qquad (1)$$

exists. The T-invariant function F equals the L_2-projection of the function f to the subspace of T-invariant functions, and moreover, $\int F d\mu = \int f d\mu$.

Note that for *ergodic* systems, that is, for those where all invariant functions are constant almost everywhere, the last claim of the theorem is just the equality of (asymptotic) time and space averages.

Shortly after von Neumann discussed this result with G. D. Birkhoff, the latter established his *individual* ergodic theorem (G. D. Birkhoff, Proof of the ergodic theorem, *Proc. Nat. Acad. Sci.* vol. 17 (1931) pp. 656-660.) stating that in (1) the convergence also holds almost everywhere. (For a more detailed history of the first ergodic theorems see G. D. Birkhoff and B. O. Koopman, Recent contributions to the ergodic theory, *Proc. Nat. Acad. Sci.* vol. 18 (1932) pp. 279-282.) Von Neumann was also aware of the L_2 form of his theorem: if U is any unitary operator in a Hilbert space H, then the averages $\frac{1}{n}(f + Uf + \cdots + U^{n-1}f)$ converge for every $f \in H$.

Von Neumann's Minimax Theorem

András Simonovits

Von Neumann made a number of outstanding contributions to game theory, most notably the minimax theorem of game theory, a linear input-output model of an expanding economy, and the introduction of the von Neumann-Morgenstern utility function. Here I present the first of these.

There are two players, 1 and 2, each has a finite number of pure strategies: $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$. If player 1 chooses strategy s_i and player 2 strategy t_j, then the first one gets payoff u_{ij} and the second one $-u_{ij}$. To hide their true intentions, each randomizes his behavior by choosing strategies s_i and t_j independently with probabilities p_i and q_j, respectively. For random strategies $p = (p_i)$ and $q = (q_j)$, the payoffs are the expected values $u(p, q) = \sum_{i=1}^m \sum_{j=1}^n p_i q_j u_{ij}$ and $-u(p, q)$, respectively. An equilibrium pair is defined as (p, q) such that

$$\min_q \max_p u(p, q) = \max_p \min_q u(p, q).$$

Theorem 1 (*von Neumann 1928*) *For any two-player, zero-sum matrix game, there exists at least one equilibrium pair of strategies.*

Remarks

1. Although another mathematical genius, Émile Borel, studied such games before von Neumann, he was uncertain if the minimax theorem holds or not.
2. This theorem was generalized by Nash (1951) (Nobel Prize in Economics, 1994) for any finite number of players with arbitrary payoff matrices (J. Nash, "Non-Cooperative Games," *Annals of Mathematics 54*, (1951) pp. 289–295.).

Example. Let us consider the following game. Players 1 and 2 play Head (H) or Tail (T) in the following way. Each puts a coin (say a silver dollar) on a table simultaneously and independently. If the result is either HH or TT, then 1 receives the coin of 2, otherwise 2 receives the coin of 1. There is a single equilibrium: $p_i^* = 1/2 = q_j^*$, which can be achieved by simply throwing fair coins.

http://www.econ.core.hu/english/inst/simonov.html

Moving to the US

Moving to the US in 1930–1931 apparently did not make a big change in von Neumann's mathematics. For a time he did not face challenges such as encountering the revolution of quantum physics in Göttingen. It is known that he had trouble getting used to the higher publication-centeredness of the American mathematical community, and the lower level of informal communication as compared with his European experiences. The famous parties of the von Neumann family in Princeton, offering beyond alcoholic drinks both mathematical and intellectual communication, partly solved this problem.

Events of his private life may have influenced his work. In the years around his divorce and second marriage, his productivity was less than his average (in 1938 he published just one work, and in 1939 none), but this is not essential. The proximity and then the beginning of the war,

Johnny during his years in the United States. (Photo by Alan Richards, Fuld Hall in Institute for Advanced Study, Princeton)

however, brought a dramatic change in von Neumann's mathematical interests and perhaps in his style. His hatred for Nazi Germany and his engagement in promoting victory over it had certainly pushed him toward essentially new questions. These were connected to new fields and new problems whose solution he hoped would contribute to the big goal. At the same time, these problems offered new mathematical challenges.

War Effort: Ballistics and Shocks

Ballistics. From the technological point of view, the war after the First World War was its continuation; the important work on improving firing tables of shells did not cease. (During the Second World War Kolmogorov also participated in similar work; see B. Booss-Bavnbek and J. Høyrup: "Mathematics and War: an Invitation to Revisit," *Math. Intelligencer* 25 (2003), no. 3, 12-25). Oswald Veblen, who became the first professor of the Princeton Institute for Advanced Study in 1932, and was responsible for inviting von Neumann to Princeton and later to the Institute, had been the commanding officer of this research between 1917 and 1919 in the US Army Ordnance Office at Aberdeen Proving Grounds, Maryland. (Among others, the great mathematicians J. W. Alexander and H. C. M. Morse also worked there under his guidance.) In 1937, von Neumann, who from his knowledge of history and from experience had already expected a war in Europe, decided to join the work in Aberdeen. Details of his progressive involvement are again very nicely described in Macrae's book, and I restrain myself from repetitions. The essential thing is that—because of the decreasing density of air with altitude—the equations governing the trajectories are nonlinear and cannot be solved exactly and even have new types of solutions. Later, especially during his 1943 visit to England, von Neumann also became interested in the dynamics of magnetic mines. His first publications about shock waves were i) an informal progress report to the National Defense Committee, which dates back to 1941, and ii) its detailed version [vN43]. As usual in von Neumann's mathematics, this very first work already gave a deep and broad overview, proved interesting new results, and provided a program, too. This report was then followed by an analogous report on detonation waves in 1942, and by further works in 1943 (3), 1945 (2), 1947 (1), 1948 (1), 1950 (1), 1951 (1), and 1955 (1). (A concise and nice description of von Neumann's approach and results is given by Fritz in [BV].)

Meteorology and Hydrodynamics. One of von Neumann's favorite subjects was meteorology. He was quite optimistic that with progress in understanding the nature of aero- and hydrodynamic equations and of their solutions, and with the development of computation, a practically satisfactory weather forecast would become possible. His hopes were only partially realized. He was certainly aware of the chaotic nature of the solutions of hydrodynamic equations, but he probably did not see sufficiently clearly the limitations caused by chaos (one should not forget that E. Lorentz's revolutionary work [Lor] appeared only in 1963, and its import was not appreciated for quite a while after that).

The study of trajectories of shells, his study of the dynamics of magnetic mines in Britain in 1943, the work on hydrodynamics equations, and later his participation from 1943 in the Manhattan project, all of these projects brought home to him the fundamental importance of computations, and consequently that of the development of computers. His Memorandum written in March 1945 to O. Veblen on the use of variational methods in hydrodynamics [vN45b] starts with the sentence "Numerical calculations play a very great role in hydrodynamics."

Computers: Neumann-Architecture and Scientific Computing

If the applied problems mentioned required a tremendous amount of computations, this appeared most spectacularly in Los Alamos, where for the work on the atomic bomb it was absolutely essential to replace experiments by mathematical modeling requiring a lot of computations. It is well known that in August 1944, on the railroad platform of Aberdeen, von Neumann by chance met Herman Goldstine, a main figure working in Philadelphia on the development of ENIAC. He immediately received a first lesson on the actual level of progress, soon got into the work, and presented groundbreaking new ideas (his first fundamental works were [vN45a] and [BGvN46]). So he not only became a devoted advocate of the project, but substantially participated in it in various ways; his two most important contributions were, first, the elaboration of the principles of the programmable computer, of the so-called von Neumann-computers, and second, the implementation of these principles in the construction of the computer IAS to be built at the Princeton Institute. This whole story is described in Aspray [A] and Legendi-Szentiványi [LSz], for instance. Let me just mention some key words about von Neumann's main implementations: RAM, parallel computations, flow-diagrams, program libraries, subroutines. I would say that if problems of hydrodynamics and of the construction of computers were a necessity and were in the air, so to speak, the importance of the use of computers for scientific research was not equally obvious. Von Neumann

On Reliable Computation

Peter Gács

Von Neumann's best-known contribution to computer science is the conceptual design of general-purpose computers. Ideas of a stored-program computer had already been circulating at the time among engineers, but von Neumann's familiarity with the concept of a universal Turing machine helped the clean formulations in John von Neumann's "First draft of a report on the EDVAC" (Technical report, University of Pennsylvania Moore School of Engineering, Philadelphia, 1945). Control unit, arithmetic unit, and memory (which also holds the program) still form the major organizational division of most computers. The arithmetic and control units are modelled well as logic circuits with connections such as "and", "or", "not", and bit memory (von Neumann called these "artificial neurons" after McCulloch and Pitts).

The report mentions the problem of physical errors, without proposing any solution. Von Neumann returned to the error correction problem in "Probabilistic logics and the synthesis of reliable organisms from unreliable components" (in C. Shannon and McCarthy (eds.), *Automata Studies*. Princeton University Press, Princeton, NJ, 1956). The most important theorem of this paper rich in ideas can be formulated as follows. Call a logic component *ideal* if it functions perfectly, and *real* if it malfunctions with probability, say, 10^{-6}. Suppose that a given logic circuit C is built with N ideal components. Then one can build out of $O(N \log N)$ real components another logic circuit C' that computes, from every input, the same output as C, with probability ≥ 0.99.

In the solution, some parts (the so-called "restoring organs") of the circuit C' are built using a random permutation. This method was rather new at the time, though Erdős's random graphs and Shannon's random codes existed already. The proof of the main theorem is somewhat sketchy: a rigorous proof appeared almost 20 years later (by Dobrushin and Ortyukov, using somewhat different constructions).

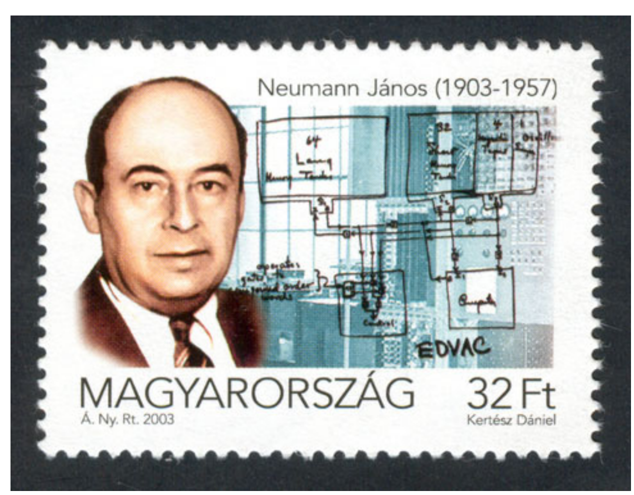
There have been some more advances since (for example, making the circuits completely constructive with the help of "expanders"), but the log N redundancy has not been improved in the general case. For a recent publication containing references see Andrei Romashchenko, "Reliable computation based on locally decodable codes" (in *Proc. of STACS*, LNCS 3884, (2006) pp. 537–548).

Error-correcting computation, in the general setting introduced by von Neumann, has not seen significant industrial applications. Surprisingly, the decrease in the size of basic computing components (now transistors on a chip) has been accompanied with an increase in their reliability: the error rate now is one in maybe 10^{20} executions. This trend has physical limits, however. (Von Neumann himself contemplated these limits in a lecture in 1949. He was not the first one, Szilárd probably preceded him, and the nature of these limits has been a subject of lively discussion among physicists ever since.) Restricting attention to logic circuits sidesteps the problem of errors in the memory and the issue of the bottleneck between memory and control unit. There are parallel computing models without division between memory and computation: the simplest of these, cellular automata, was first proposed (for other purposes) by von Neumann and also by Ulam. By now, universal reliable cellular automata have also been constructed.

http://www.cs.bu.edu/~gacs

saw this very quickly, his brain was completely prepared for it, and he devoted a substantial amount of energy to it.

Reading his papers on this theme, one feels that he was not only aware of the great perspectives that computers had opened up, he was at the same time discovering and enjoying the new horizons with a lively, childish curiosity. A simple "game" was, for instance, to calculate the first 2000 decimal digits of e and of π by the ENIAC in 1950. Aspray discusses von Neumann's accomplishments in scientific computing in detail. For the record I add that about the principles of construction of computers he had reports in 1945 (1), 1946 (1), 1947 (1), 1948 (1), 1951 (2), 1954 (2), and 1963 (1). On scientific computing he had the first paper, joint with V. Bargmann and D. Montgomery [BMvN46], in 1946 (1), on solving large systems of linear equations by computers. It was followed by works in 1947 (4), 1950 (3), 1951 (2), 1953 (1), 1954 (1), and 1963 (2). He made computer experiments in number theory, ergodic theory, and stellar astronomy. He also had several suggestions for random-number generators (via the middle-square method or the logistic map), and, with Ulam, he was also working on inventing the Monte-Carlo method. (I emphasize that the publication counts are only for providing a feeling. The topic of a work is sometimes ambiguous; some manuscripts only appeared after his death; and some reports may have appeared as articles (I have not checked overlap).) Had he lived longer, he certainly would have greeted the amalgamation of computation with science and would have



Commemoration of the creation of the EDVAC computer. (Credit: http://www.computer-stamps.com/country/22/Hungary/)

been one of the leaders. This is certainly true for computer-assisted proofs. He would have been glad to see the birth of computational sciences: computer science, computational physics, chemistry, etc. He perhaps would not have been so glad to see that often computations replace creative and logical thinking, but this is only a guess.

Last, but not least, I want to turn to a very intriguing interest of von Neumann. Having analyzed the functioning of computers, he formulated several related questions. He initiated research on cellular automata and probabilistic automata (i.e., automata with unreliable components). He raised the question of constructing self-correcting automata. Moreover, he envisioned the need to compare the functioning of computers with that of our brains. His articles on the subject appeared in 1958 in the collection [vN58], and the analysis of his thoughts on the topic would deserve a separate discussion.

Epilogue

All in all, fate, often combined with his own decisions, put before John von Neumann the most extraordinary and diverse scientific challenges. In all cases, he did his job in an ingenious and superb way. Factors such as his genes, family, education, the intellectual ferment that surrounded him in Budapest, and the intellectual level of his teachers and of Hungarian mathematics of the time, gave him an excellent launch. Besides his intelligence, knowledge, deductive power, and fantastic mental speed, he showed depth, perspective, and taste; his diverse scientific and human interests, complemented by his education in chemical engineering, contributed to his unrivalled courage, openness, and flexibility. By excelling in an unusually broad spectrum of mathematical activities, he became an outstanding representative of twentieth-century mathematics, whose influence was unbelievably wide within and outside mathematics. Von Neumann's achievements demonstrate convincingly the strength of the mathematical approach, "unreasonable effectiveness of mathematics," as Wigner put it. It is not overstatement to compare the achievements of John von Neumann to those of Archimedes, Newton, Euler, or Gauss.

Von Neumann was exceptionally widely known among mathematicians, and there are plenty of anecdotes related to him. I think that as a student, I heard from my professor A. Rényi the saying: "Other mathematicians prove what they can, Neumann what he wants." (It fits, though I have mentioned that he was not happy that it was not himself who found Gödel's incompleteness theorem or Haar's construction of the invariant measure of a locally compact group or Birkhoff's proof of the individual ergodic theorem.)

Not long ago, Dan Stroock from MIT mentioned to me the following at least half-serious opinion: "A genius either does things better than other people, or does them completely differently (orthogonally) from other people." I think it is true about von Neumann that he did most things better than other mathematicians would have done. An example of an "orthogonal" scholar could well be Einstein. Nevertheless, I think of times when von Neumann also did things orthogonally.

Let me finish by citing a well-known anecdote. Question to Wigner: "How is it that Hungary produced so many geniuses in the early 20th century?" Wigner's answer: "That many geniuses? I don't understand the question. There was only one genius: John von Neumann."

Brief Biography

1903, December 28: Born in Budapest
1914-1921: Lutheran Gymnasium, Budapest (Teacher of maths: László Rátz. Tutored by Gábor Szego; afterward by M. Fekete and L. Fejér.)
1921 Winter term - 1923 Summer term: Berlin University, student in chemistry, attends lectures in physics and mathematics (Professor of Mathematics: Erhard Schmidt, Professor of Chemistry: Nobel Laureate Fritz Haber).
1923 Summer term - 1926 Summer term: ETH, Zürich, student in chemical engineering, attends lectures in physics and mathematics, too (Hermann Weyl, György Pólya); **October 1926**: Diploma in chemical engineering.
1921 Winter term - 1925 Summer term: registered student of mathematics at Budapest University; **March 1926**: Ph.D. in mathematics (Supervisor: L. Fejér?).
1926 Fall term - 1927 Summer term: visiting Göttingen University, grant from Rockefeller Foundation (D. Hilbert, L. Nordheim, R. Courant, K. Friedrichs, P. Jordan, future Nobel Laureate physicists Max Born, J. Franck, W. Pauli, W. Heisenberg).
1927 Fall - 1929: Privatdozent at Berlin University.
1929: Privatdozent at Hamburg University.
1930 - 1938: Marriage with Marietta Kövesi.
1930: Visiting Lecturer at Princeton University.
1931 - 1933: Visiting Professor at Princeton University.
1933 - 1957: Professor, Member of the Institute for Advanced Study, Princeton.
1935: Daughter Marina von Neumann born.
1938: Marriage with Klári Dán.
1957, February 8: Dies in Walter Reed Hospital, Washington, buried in Princeton Cemetery

A complete bibliography of John von Neumann can be found on the Internet: http://www.info.omikk.bme.hu/tudomany/neumann/javnbibl.htm. Most papers, and manuscripts are in [T], and some in [BV].

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