Chapter 2: Review of Dynamic Relaxation with an extension to six degrees of freedom theory

2.1. Introduction

This chapter reviews the development of the Dynamic Relaxation (DR) method as applied to the form-finding and load analysis of non-linear structures. The scope of the literature review is to firstly describe the initial general development of the DR process and secondly to focus on research that has direct relevance to the form-finding and load analysis of stressed spline structures. The latter is achieved by giving an outline of the principal mechanisms behind DR as applied to non-linear tension structures, and the controls needed to make them into an automated and efficient procedure. This DR procedure forms the basis of the three degrees of freedom spline theory developed in Chapter 3. Pre-stressed membrane elements, strut elements and cables are reviewed for their inclusion in the DR process as they will be incorporated in the applications developed in Chapter 5. A full account is also given of the six degrees of freedom theory for beam elements, to be utilised for comparative numerical test cases in Chapter 4.

2.2. Dynamic Relaxation

2.2.1. Review of Dynamic Relaxation

Day (1965) derived the concept of DR as an explicit solution method for the static behaviour of structures from an analogy with tidal flow computations, which he had previously worked on with Otter (1960). Equations of damped structural motion and the constitutive equations of elasticity substituted respectively the equations of fluid motion and continuity. The static solution of structural problems was then considered as the limiting equilibrium condition of heavily damped structural vibrations. Although Newmark (1959) had earlier suggested obtaining static solutions using this method, his implicit integration scheme put the technique at a disadvantage compared with standard matrix methods for non-linear analysis and the DR method in explicit form (Lynch, Kelsey and Saxe (1968)). Initially DR with both spatial and temporal finite difference idealization, was used to solve linear elastic problems including pressure vessels (Otter (1965)), cylindrical arch dams (Otter (1966)), thin plates
(Day (1966)) and thin shells (Cassell (1968)). While researching local stresses in pre-stressed concrete end blocks up to cracking, Holland (1967) firstly introduced non-linear material effects in the DR method. Methods for determining the damping factor used for the reduction in amplitude of the vibrating system were presented by Rushton (1968a). Rushton (1968b) and (1969) used DR extensively for the analysis of elastic plates under a variety of load and boundary conditions. Although Day (1965) conceived DR for solving continuum differential equations with difficult boundary conditions, he significantly stated that the technique could be applied to problems requiring a finite element type of solution. This notion was taken up by Lynch et al. (1968) who applied DR to two dimensional finite element analysis using an overall stiffness matrix. Brew and Brotton (1971) developed a DR formulation that separated equations for equilibrium and compatibility (or motion) and did not require the formulation of an overall stiffness matrix. This vector form of DR has become most widely used, particularly for highly non-linear structures and is adopted in this thesis.

In 1969 Bunce and Day described how the technique could be applied to the analysis of hanging roofs and mechanisms using finite elements. In 1970 they gave the earliest application of DR to tension structures (Bunce and Day (1970)). A centrally positioned load was applied to a plane cable net, and the resulting deformed shape was obtained using the DR method. Between the period of 1971 and 1977 Barnes used DR extensively for the form-finding and analysis of a wide range of cable networks, membranes and pneumatic structures, including the modelling of the effects of cable slackening, membrane buckling and non-linear material properties. Literature concerning other applications of the DR method to tension structures can be found in work by Papadrakakis (1978) and Wakefield (1980) to form optimisation of space structures by Topping (1978) and to tensegrity structures by Motro (1983) and Motro, Belacem and Vassart (1994). More recent reviews and applications to tension structures are given by Barnes (1994) and (1999), Lewis and Gosling (1993) and Lewis and Lewis (1996) and Wakefield (1999).

The basis of the method is to follow the movement of each node of the structure from its initial unloaded position for small time intervals $\Delta t$ until all vibrations have died out due to artificial damping. Starting from an inaccurate and arbitrary
specified geometry in the form-finding process, movement of the nodes is caused by specifying stresses in the structural elements. For the load analysis, which must start from the pre-stressed equilibrium state, the motion is caused by suddenly applying the loading.

2.2.2. Viscous Damping

Usually the finite element form of DR is described as being the means by which any unstable discretised system may be brought to rest through the application of viscous damping of the nodal movements. These movements are proportional to the product of nodal velocities and mass components. In order to achieve the most rapid convergence, the lowest mode of vibration of the structure is critically damped and the fictitious nodal mass components are adjusted to be proportional to corresponding direct stiffness components (Barnes (1976)). If critical damping is used, monotonic convergence occurs as shown in figure 2.1. From a trial analysis (i.e. un-damped or lightly damped), the fundamental frequency $f$ of the structure can be found from the displacement-time trace; an estimate of the critical damping constant is then given by:

$$C_{\text{crit}} = 4\pi\Delta t$$

(2.1.)

![Figure 2.1. Displacement - Time Curve](image)
In some cases the critical viscous damping factor might be difficult to estimate. Starting from a rather inaccurate initial position, certain elastic members are grossly deformed in the initial stages and induce locally unbalanced forces and related high-frequency modes. Therefore additional control measures such as varying mass components, locally varying damping constants and fictitious member stiffnesses are needed in different stages of the analysis to obtain convergence (Barnes (1977)). Frieze, Hobbs and Dowling (1978) used similar variable controls for the investigation of plate buckling; significantly different levels of damping were needed for varying levels of applied loading.

More recently Zhang and Yu (1989) have presented a modified adaptive Dynamic Relaxation Method based on the viscous damping approach. In this case, the damping coefficient is based on a function of the current system configuration, the internal element force and the mass matrix. When applied to the analyses of elastic-plastic plate bending, and to the wrinkling of circular plates, it was shown that this method is stable and displayed a small increase in computational efficiency, when compared with a standard viscously damped DR algorithm.

If DR is to be generally and readily used for non linear structural analysis, a method of iteration control is required that does not incorporate trial analyses for parameters and naturally copes with gross deformations and high residual forces whilst retaining the essential simplicity of the method.

2.2.3. Kinetic Damping

Cundall (1976) working on unstable geo-mechanical problems, suggested using kinetic damping which proved to be entirely stable and rapidly converging when dealing with large unbalanced local forces (Barnes, Topping and Wakefield (1977)). As viscous damping is neglected, there is no need for prior determination of the damping constant. The underlying basis of kinetic damping is that as an oscillating body passes through a minimum potential energy state, its total kinetic energy reaches a local maximum. Under this scheme, the total kinetic energy is traced under un-damped motion of the structure. In a linear elastic single degree of freedom system, the first peak of kinetic energy is achieved as the structure passes through its static equilibrium position when all
nodal velocities are maximum. However, in a multi-degree of freedom system, which applies to virtually all structures, the structure exhibits un-damped vibrations in a combination of all its modes. Upon detection of a local energy peak, all current nodal velocities are set to zero. The process is then restarted from the current geometry and continued through generally decreasing peaks until the energy of all modes of vibration has been dissipated and the structure attains its static equilibrium state. A typical kinetic energy trace is shown in figure 2.2. for the case of form-finding of a cable net or membrane with inaccurate initial geometry. The early energy peaks (1) are associated with high frequency modes caused by large out-of-balance forces in boundary or mast support regions. After these modes have been substantially damped out, subsequent peaks (2) are associated with the overall structural form and the lowest frequency modes, the motion in these modes being generally normal to the changing surface. Near convergence, low energy peaks (3) occur rapidly, associated with slight and high frequency in plane motion.

![Figure 2.2. Dissipation of kinetic energy](image)

For form-finding and load analysis of path dependent problems involving both geometric and material non-linearities (e.g. cable and membrane slackening and spline buckling), a combination of viscous and kinetic damping in the DR process might be advantageous. In this case the overall behaviour can be damped by the viscous term, with high frequency local disturbances (on which viscous damping has little effect) being dissipated by reinitialising at kinetic energy peaks. For the sake of simplicity the following description of DR therefore
contains both viscous and kinetic damping terms. Literature concerning the application of both viscous and kinetic damping to the form-finding of tension structures can be found in the work by Lewis, Jones and Rushton (1984).
2.2.4. Basic Equations

A numerical model of the structural system is built as a set of nodes with lumped masses connected by elements. Subsequently the equilibrium of each node is examined in terms of the balance of applied loads and forces of attraction and repulsion in the adjoining elements.

The governing equation in the DR method is Newton’s second law of motion:

\[ \text{Force} = \text{Mass} \times \text{Acceleration} \]  

(2.2.)

This law is rewritten for the displacement of any node \( i \) in the x-direction at time \( t \) as:

\[ P_{ix} - K_{ix} \delta_{ix} - C_{i} v_{ix} = M_{i} \dot{v}_{ix} \]  

(2.3.)

Where:

- \( P_{ix} \): Applied force at node \( i \) in direction \( x \)
- \( K_{ix} \): Stiffness term at node \( i \) in direction \( x \)
- \( \delta_{ix} \): Total displacement of node \( i \) in direction \( x \) at time \( t \)
- \( C_{i} \): Viscous Damping constant at node \( i \)
- \( v_{ix} \): Velocity of node \( i \) in direction \( x \) at time \( t \)
- \( M_{i} \): Lumped fictitious mass at node \( i \) chosen to optimise convergence
- \( \dot{v}_{ix} \): Acceleration at node \( i \) in direction \( x \) at time \( t \)

Note that if only kinetic damping is applied to the DR then the viscous damping component in equation (2.3.) can be ignored.

The relationships given in the equation 2.3. apply equally for displacements of any node \( i \) in the y- and z-directions at time \( t \). In figure 2.3. equation 2.3. is visualised as a Spring-Mass-Damper System for a one degree of freedom system.
Equation 2.3. can be rewritten for any time $t$ as:

$$R_{ix}^t = M_i \ddot{v}_{ix}^t + C_i \dot{v}_{ix}^t$$

(2.4.)

Where:

- $R_{ix}^t$ Residual (or resultant) of the applied and structural member forces at node $i$ in direction $x$ at time $t$

Expressing equation 2.4. in central finite difference form for a small time step $\Delta t$ gives:

$$R_{ix}^t = M_i \left( \frac{v_{ix}^{t+\Delta t/2} - v_{ix}^{t-\Delta t/2}}{\Delta t} \right) + C_i \left( \frac{v_{ix}^{t+\Delta t/2} + v_{ix}^{t-\Delta t/2}}{2} \right)$$

(2.5.)

Replacing $C_i$ by $C \cdot M_i$, with damping thus proportional to mass, equation 2.5. can be re-arranged to give a recurrence equation for the nodal velocity at time $t + \Delta t/2$:

$$v_{ix}^{t+\Delta t/2} = \frac{\Delta t}{M_i} R_{ix}^t + B v_{ix}^{t-\Delta t/2}$$
Where:

\[
A = \frac{1}{1 + C \Delta t/2}
\]  
(2.6.)

\[
B = \frac{1 - C \Delta t/2}{1 + C \Delta t/2}
\]  
(2.7.)

\[C\] Damping factor per unit mass

Whence the updated geometry projected to time \((t + \Delta t)\) is

\[
x_i^{t+\Delta t} = x_i^t + \Delta v_i^{t+\Delta t/2}
\]  
(2.9.)

Where:

\[x_i^{t+\Delta t}\] \(x\) co-ordinate at node \(i\) at time \(t+\Delta t\)

Using equations 2.6. and 2.9. the velocity components and co-ordinates in all three directions can be similarly updated for all nodes of the structure. The equations are nodally de-coupled in the sense that they depend only on a node’s own previous velocity and residual force components. The current \((t + \Delta t/2)\) updates of other nodes do not affect the solution. From the updated node co-ordinates, the new link forces can be found and resolved with the applied load components to give the updated residual forces.

\[
R_i^{t+\Delta t} = P_i + \sum \left( \frac{F}{L} \right)_m (x_j^t - x_i^t)^{t+\Delta t}
\]  
(2.10.)

where the summation is for all links \(m\) connecting node \(i\) to adjacent nodes \(j\)
Where:

\[ F_{m}^{t+\Delta t} \quad \text{Current} \ (t + \Delta t) \ \text{force in link} \ m \ \text{connecting node} \ i \ \text{to adjacent nodes} \ j \]

\[ L_{m}^{t+\Delta t} \quad \text{Current length of link} \ m \]

Equation 2.9. containing the compatibility condition, is separated from the equilibrium equation 2.10. which is characteristic of an explicit vector scheme such as DR.

2.2.5. Summary of the iterative computation procedure

The cycle of calculations for dynamic relaxation with kinetic and viscous damping can thus be summarised as follows:

1. Set all residual forces of the nodes, all nodal velocities and the (previous) kinetic energy to zero.

2. Set the residuals equal to all the applied load components.

3. Calculate the link forces from the current element stresses and add their resolved components to the residuals (equation 2.10.).

4. All residuals of fixed or partially constrained nodes are reset to zero. This ensures they do not move in the constrained direction.

5. Calculate all nodal velocities and hence the updated geometry (equations 2.6. and 2.9.). Determine the current kinetic energy of the entire system which is proportional to

\[ \sum M_i \sum \nu_{ix}^2 \ \text{at} \ t + \Delta t/2 \ \text{for all nodes} \ i \ \text{and directions} \ x = 1:3 \]

(2.11.)

If the current kinetic energy is not a local peak, the process is repeated from step 2. However if a peak in kinetic energy has been detected, the process is
restarted at step 1. On restarting the Dynamic Relaxation Process, all residual
nodal forces, the previous kinetic energy and all nodal velocities are reset to zero.
The velocity of node $i$ in direction $x$ at the first midpoint time interval is then
defined below to give effectively $v_{ix}^{\Delta t/2} = -v_{ix}^{-\Delta t/2}$ or $v_{ix} = 0$ at time zero.

$$v_{ix}^{\Delta t/2} = \frac{\Delta t}{2 M_i} R_{ix}^0$$

(2.12.)

Additionally, when a peak in kinetic energy is detected, the co-ordinates
will have been projected to a time $t+\Delta t$ whereas the “true” kinetic energy peak
co-ordinates will have to be found at some earlier peak time $t^*$. The updated co-
ordinates projected to time $t+\Delta t$ are based on average velocities at mid-points of
time intervals. So that

$$x_{i}^{t+\Delta t} = x_i^t + \Delta t v_{ix}^{t+\Delta t/2}$$

(2.13.)

Barnes and Wakefield (1984) showed that these true kinetic energy peak
co-ordinates can be found using

$$x_i^{t^*} = x_i^{t+\Delta t} - \Delta t (1 + q) v_{ix}^{t+\Delta t/2} + \frac{\Delta t^2}{2} q \frac{R_{ix}'}{M_i}$$

(2.14.)

Where:

$$q = \frac{B - C}{2B - C - A}$$

with terms A, B and C defined in figure 2.4.. Alternatively, the peak can be assumed to occur at $(t - \Delta t / 2)$ and hence
$A = C$ and $q = 1/2$ in equation 2.14.. This is the same as taking the velocities
during the intervals $(t - \Delta t \rightarrow t)$ and $(t \rightarrow t + \Delta t)$ as constants equal to the mid-
point values.
Figure 2.4. Terms A, B and C on Kinetic Energy versus Time diagram

Once the vibrations of the structure around the equilibrium position are satisfactorily small and the residual forces at all nodes and the current kinetic energy is low, the analysis can be stopped since static equilibrium is practically achieved.

2.2.6. Numerical Stability

The scheme will become unstable when the time interval $\Delta t$ exceeds a certain critical value or when the fictitious masses of the nodes are too low. The critical situation of a group of elements and nodes all aligned in their highest frequency mode governs numerical stability. The highest frequency mode occurs if all nodes $i$ move in relative opposition to their adjacent nodes $j$ (see figure 2.5.). An overall motion might be imposed, associated with a lower frequency mode, but this has no effect on the numerical stability.

Figure 2.5. Nodes $i$ moving in relative opposition to nodes $j$
To examine the most critical condition that can arise the following assumptions are made:

1. The nodes in the group all have the same fictitious mass $M$.

2. The direct stiffness $S$, of any node relative to adjacent nodes and in the highest stiffness direction, is identical for all nodes in the group.

3. The motion of the nodes is parallel to the principal stiffness direction.

4. All nodes $i$ have identical velocities $v_i$ and all nodes $j$ have identical velocities $v_j$

Having made these assumptions to describe the most critical condition, a limit can be put on the time interval $\Delta t$ or on the fictitious mass $M$ to ensure convergence of the iteration process. Barnes (1980) derives a limit for the time interval $\Delta t$ for a lumped nodal mass scheme:

$$\Delta t = \sqrt{\frac{2M}{S}}$$

(2.15.)

Where:

$$\frac{M}{S} \quad \text{Smallest ratio of nodal mass to direct stiffness at any node}$$

If the intention of the analysis is to trace the dynamic behaviour using ‘real’ lumped nodal masses then equation 2.15. would give the greatest value of $\Delta t$ that can be used to ensure stability, provided the node with the least ratio (mass/principal direct stiffness) was chosen for $M/S$. For other nodes, which are not aligned with the principal stiffness direction, the condition is less critical justifying the assumptions made above. For analyses kinetically damped to provide a static solution, however, the masses may be fictitious and adjusted to
optimise the rate of convergence. For any arbitrary time interval \( \Delta t \), the fictitious mass at any node is thus set as

\[
M_i = \frac{\Delta t^2}{2} S_i
\]

(2.16.)

Where:

\( S_i \)  Greatest direct stiffness that can occur at node \( i \) during the form-finding or load analysis processes

This may not be its initial principal stiffness if the structure is undergoing gross deformations. Of course it is an oversimplification to assume that only one lumped mass value (related to the maximum principal stiffness), is used for each node. However it has been found (Wakefield (1980)) that there is no great advantage to be gained by relating the component stiffnesses in all local principal directions at each node to three nodal mass components. A single component of mass is therefore used in this thesis.

2.2.7. Pinned Skeletal Structures

In cable-braced stressed spline structural systems (or skeletal space structures generally), the link force used in equation 2.10. is given by

\[
F^{s+s}_{m} = F^{s}_{m} + K^{s}_{m}(L^{s+\Delta t}_{m} - L^{s}_{m})
\]

(2.17.)

Where:

\( F^{s}_{m} \)  Initial or specified force in link \( m \)
\( K^{s}_{m} \)  Initial or specified Elastic Stiffness of link \( m \)
\( L^{s}_{m} \)  Initial or specified length of link \( m \) at force \( F^{s}_{m} \)

If the current force from equation 2.17. is evaluated as negative for a cable link, it is set as zero; this check is carried out at every iteration. The elastic stiffness \( K^{s}_{m} \) for links is defined as
\[ K_m' = \left( \frac{EA'}{L_0} \right)_m \]

(2.18.)

Where:

- \( E \) Young’s Modulus
- \( A' \) Specified (or effective) cross-sectional area

The factors \( F_m', L_m', K_m' \) (for unbuckled elements) are specified physical constants during load analysis, but in form-finding they may all be used to control the form. In the form-finding process of geodesic nets, for example, in which cables follow paths of minimum distance over the surface, the elastic stiffnesses \( K' \) are set to zero. The tension throughout each cable is held constant, while the cable links slide over each other and grossly change length. Concerning the stability and the setting of mass components in the form-finding of a geodesic net, the greatest direct stiffness at the nodes is therefore solely the ‘geometric stiffness’:

\[ S = G_1 \sum \frac{T'}{L'} \]  

(for all members meeting at a node \( i \))

(2.19.)

Where:

- \( G_1 \) Factor that allows for the increase of initial geometric stiffness due to possible gross shortening of link lengths during the process
- \( T' \) Specified tension

However in the form-finding process of an orthogonal uniform mesh net, the link lengths are all set equal and \( K_m' \) are the real elastic stiffnesses. In this case, the form is controlled by only a few links (such as the end links adjoining boundary scallops) either by holding their tensions constant (\( K_m' = 0 \)) or by using very low stiffness and adjusting their initial lengths \( L_m' \). This procedure is
obviously only possible if the initial connectivity of the system is feasible and the interior mesh of links will fit across the surface without some of them being slack or end links being grossly extended. For the analysis of an orthogonal uniform mesh net, the greatest direct stiffness at the interior nodes, used for setting the mass components, is aligned with one of the traverses and is given by

\[ S = \sum \frac{EA^s}{L_0} \]  

(for two consecutive links)

\[(2.20.)\]

Where:

- \( L_0 \): Initial length of member

More generally, both elastic and geometric stiffness must be accounted for as well as very large changes in the geometry and orientation of elements, particularly during form-finding from a crude initial state. Thus to ensure stability at every node \( i \) the masses \( M_i \) can be set according to equation 2.16. with:

\[ S_i = \sum \left[ \frac{EA^s}{L_0} + G_1 \frac{T^s}{L^s} \right] m \]  

\[(m \text{ all members meeting at node } i)\]

\[(2.21.)\]

The greatest possible direct stiffness \( S_i \) (see figure 2.6.) defined in equation 2.21. covers the worst possible case of all links at the node becoming aligned in a single direction during any stage of the analysis process. The greatest direct stiffness \( S_i \) means the greatest stiffness relative to the adjacent nodes, which are considered fixed.
Figure 2.6. $S_i$, greatest stiffness for node $i$
2.2.8. Pre-stressed Membrane Elements

A coated fabric membrane surface consists of a series of fabric panels, which are sewn or welded together along common seams between panels. The warp direction of the fabric weave is aligned with the centreline of the panels and ideally follows a geodesic path. As the seams will be nearly parallel with the warp fibres, they will in turn become geodesics on the surface, optimising the use of the fabric roll for the production.

In the numerical model, each panel can be modelled with a sufficiently large number of simplex (or constant strain) triangular membrane elements with one side of every element along a ‘warp (or geodesic) control line’. In form-finding, all the membrane elements are under stress control (i.e. the elastic stiffness of each element is set to zero) but the prescribed warp and weft stresses, $\sigma_x$ and $\sigma_y$ respectively, are assigned to the element referenced parallel and perpendicular to the warp control lines. These lines and the element idealisation must be found during the form-finding process.

Haug (1980) suggested a scheme of superimposing cables of uniform but very low tensile pre-stress over the membrane surface. These cables, and thus the nodes along them, will tend towards geodesic paths over the surface. The tensions in these cables have to be very small so as not to influence the overall surface form, but if they are too small they will not follow geodesic paths accurately because of the use of discrete elements to idealise the continuum surface.

Barnes and Wakefield (1984) proposed a modified form of Haug’s idea by using ‘fictitious’ warp control strings in which the tensions govern trajectories in the plane of the surface but have no effect normal to the surface. The tensions are constant along these strings and each string will therefore automatically find the shortest path between its starting node and the last node on the surface to define a geodesic. At end nodes, for example on boundary cables, the effect of strings must also be entirely disregarded. The form of the surface is thus entirely dependent on the specified stresses in membrane elements which, although held constant in individual elements, may be graded throughout the entire surface. The tension in the warp control strings is most conveniently controlled by
specifying constant tension coefficients \( (T/L) \) in their component links. A tension coefficient is the ratio of the tension to the length between two adjacent nodes of the string. The following computational sequence was proposed as a means of imposing the above mentioned conditions on the warp control strings:

1. Set the nodal residuals \( \{R\} \) due only to control warp strings before any other element type.

2. Calculate the average value of surface normal vectors of adjacent triangular membrane elements meeting at a node. A weighting factor is applied to each normal vector, which is inversely proportional to the element area. This average value is then taken as the surface normal vector \( \{v_n\} \) at the common node.

3. The components of the nodal residual forces normal to the surface \( R_n \) are then given by \( R_n = \{v_n\}^T \{R\} \).

4. For each geodesic warp control string node, subtract the normal component from the global residuals: \( \{\overrightarrow{R}\} = \{\overrightarrow{R}\} - R_n \{v_n\} \)

5. Set the nodal residuals of all ridge and boundary cables to zero.

6. Proceed to residual summation for other element types.

In its deformed state, side 1 of the one facet membrane element is always parallel to the x-axis (or warp direction) and all the triangle sides are assumed to remain straight. In the form-finding process shear stresses \( \tau \) are never imposed in the membrane. The link attraction forces \( T_i \) between the 3 nodes of the element due to direct stresses \( \sigma_x \) and \( \sigma_y \) are derived by Barnes (1994) as:
The terms $h, l_1, l_2, l_3, \alpha_1, \alpha_2$ and $\alpha_3$ are defined in figure 2.7. for the current deformed state.

Figure 2.7. Triangular Membrane Element

In the particular case of a minimum surface with a uniform overall stress $\left(\sigma_x = \sigma_y = \sigma\right)$, the link tension coefficient along any side $i$ can be expressed as:

$$\frac{T_i}{l_i} = \frac{\sigma}{2\tan\alpha_i}$$

(2.23.)

Expressions for the tension coefficients in the case of non-standard elements (with side 1 not parallel to the warp direction) can also be developed, but are not used within this thesis.

Although the overall form, element sizes and shapes and reference axes (always belonging to the warp control lines) change continuously in the form-finding process, it is sufficient to reset the tension coefficients for membrane elements from equation 2.22. only at kinetic energy peaks. This, together with the simple calculations of equation 2.22., makes convergence of the analysis very rapid.
Concerning the numerical stability during form-finding, consider a group of elements as shown in figure 2.8. The geometric stiffness at node $i$ is:

$$\sum \frac{T}{L} = a\sigma_i \frac{2}{b} + b\sigma_j \frac{2}{a}$$

(2.24.)

Where:

$a, b$ Terms defined in figure 2.8.

This geometric stiffness must be added to the geometric stiffness of the warp control strings, which, for every node along the string, is twice the warp string tension coefficient. For nodes on boundary or ridge cables, the elastic stiffness of the cable links must be included in the same way as for cable nets.

During load analysis of membrane structures, shear stresses will be induced by applied loadings. Although the shear stiffness of coated fabrics is so low that the principal stresses will always be in the weave directions, shear terms must be included in order to prevent distortion of the element mesh. The link tensions due solely to shear stress $\tau$ are given by Barnes (1994) as:

$$T_1 = \frac{d_1}{2} - \frac{dh}{\tan \alpha_2}$$

$$T_2 = -\frac{d_2}{2}$$

$$T_3 = \frac{d_1}{2}$$
All the terms are defined in the deformed state in figure 2.7.

Both the direct and shear stresses can be related to the strains of the pre-stress state. If $l_i'$ and $h_i'$ are the length of side 1 and the perpendicular from node 1 to side 1 in the pre-stress state, the strains are:

$$\varepsilon_x = \frac{l_i}{l_i'} - 1 \quad \varepsilon_y = \frac{h_i}{h_i'} - 1$$

$$\gamma = \frac{1}{\tan \alpha_2} - \frac{1}{\tan \alpha_2} \left( \frac{1 + \varepsilon_x}{1 + \varepsilon_y} \right)$$

Where:

- $\varepsilon_x$: Strain in $x$-direction
- $\varepsilon_y$: Strain in $y$-direction
- $\gamma$: Shear Strain
- $l_i'$: Length of side 1 in pre-stress state
- $h_i'$: Perpendicular length from node 1 to side 1 in pre-stress state
- $\alpha_2$: Angle at node 2 in pre-stress state

In the case of a non-standard element, the shear term is unchanged, but the direct strains must be related to the element’s distortions parallel and perpendicular to its reference axis. The shear stress for the pre-stress state can be taken as:

$$\tau = G_2 \gamma$$

Where:

- $G_2$: Factor typically 1/20 of the lowest direct modulus
Barnes (1994) describes numerical ways to deal with crimp interchange between weft and warp and with on/off buckling or slackening of the element links. This triangular element will be used in Chapter 5 in the form-finding and analysis of spline supported membrane structures.
2.3. Six Degrees of Freedom Beam Elements

It is appropriate to give a detailed explanation of a six degrees of freedom theory, which allows the implementation of beam elements in the DR process. The method is used to provide a numerical check in Chapter 4 against the three degrees of freedom spline theory developed in Chapter 3, which forms the theoretical core of this thesis. The problem of modelling geometrically non-linear beams using DR with kinetic damping has previously attracted the attention of Wakefield (1980) and Ong (1992). In the theory they proposed, the beam consists of a series of straight two-node elements with three translational and three rotational degrees of freedom per node. These orientations of the beam cross-section are defined with reference to a third node. At kinetic energy peaks, the influence of current axial load on the moment/curvature relations is updated by the application of the $s$ and $c$ stability functions (Livesley and Chandler (1956)). This method has been successfully used in the design and analysis of the bending elements for various structural systems (Wakefield (1999)). Williams (1999) considered the shape-finding and load analysis of grid shells and developed an alternative approach, which is explained in the next sections. This does not contain new theory but assembles existing pieces into a coherent procedure for the form-finding and load analysis of spline beam elements with six degrees of freedom.

2.3.1. Transformation matrix $[A]$

This approach treats the spline non-linearly, allowing for large rotations. All the mathematical derivations of the formulae in this Chapter are given in appendix A. The spline is modelled as a series of links between nodes with six degrees of freedom (three rotations and three translations). The number of nodes on the spline is chosen in order for the node rotations relative to adjacent nodes to be small. In this way the non-linear effect of element bowing is made insignificant. There is no assumption about the size of the rotations but is assumed that the relative rotation between the two ends of an element is small. This means that a member may have to be divided into a number of sub-elements. The number of sub-elements required for a given problem can best determined by numerical experiments. The approximations occur in a number of ways, firstly by assuming
that the shape of the member is a cube and secondly by making approximations of the form \( \sin \alpha = \alpha \). Each node is defined as a point uniformly or non-uniformly spaced along the spline. Associated with each node there are two or more local orthogonal vector axes systems. Firstly there is the axis system associated with the node itself \( (\vec{x}_{\text{node}}, \vec{y}_{\text{node}}, \vec{z}_{\text{node}}) \) and secondly there is an axis system \( (\vec{x}_{\text{member, end}}, \vec{y}_{\text{member, end}}, \vec{z}_{\text{member, end}}) \) for the end of each member that is connected to that node as shown in figure 2.9. Here ‘node’ is the node number, ‘member’ is the member number and ‘end’ is 1 or 2. The \( \vec{z}_{\text{member, end}} \) is tangent to the member at its end and \( \vec{x}_{\text{member, end}} \) and \( \vec{y}_{\text{member, end}} \) are the principal bending axes. The member end axes at a node are defined as components in the nodal axis system so that as the node rotates, it forces the member ends to rotate with it.

![Figure 2.9. Member end vector axes system and nodal vector axes system](image)

The transformation matrix \( [A] \) for a particular node specifies the rotation the nodal vector axes system has undergone relative to the fixed global \( x, y, z \) axes system. In general the local axes systems at each end of a member do not have the same orientation and the member is modelled through a shape function.
The original starting geometry for the relaxation procedure may be stressed or unstressed and the structure may or may not be in equilibrium when no external loads are applied to it. The aim of the relaxation procedure is to move and rotate the nodes to achieve equilibrium. At each node the nodal axis system will move and rotate and the associated member axis system will move with it. To obtain the associated member axis rotation, the initial member axis system needs to be pre-multiplied by the orthogonal transformation matrix \([A]\).

Many papers have been written that refer to finite rotations and the associated transformation matrix \([A]\) as reviewed by Beatty (1977). On the one hand Euler (1744) is responsible for the classical origins of the theory of rigid body motion based on scalar methods. Rotation variables were suggested in the past such as Eulerian (Buechter and Ramm (1992)) and Cartesian angles. If the angles of rotations \(\alpha, \beta, \gamma\) are known (Ledermann (1961) Ramm (1976)), the transformation matrix \([A]\) can be written as

\[
[A] = \begin{bmatrix}
\cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & -\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma & \sin \alpha \sin \beta \\
\sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma & -\cos \alpha \sin \beta \\
\sin \beta \sin \gamma & \sin \beta \cos \gamma & \cos \beta 
\end{bmatrix}
\]

(2.28.)

On the other hand Williams (1999) takes a different approach by adopting a vector method introducing the concept of a rotation vector to build the transformation matrix. This method is adopted in this thesis. The next sections expand upon the transformation matrix \([A]\) based upon the idea of the rotation vector.

2.3.2. Rotation vector
Beatty (1977) introduces the idea of describing any finite rotation (or sequence of rotations) as a simple rotation of magnitude $\alpha$ about some fixed axis described by a unit vector. This concept is taken further by defining the transformation of the original vectors $(\vec{x}_o, \vec{y}_o, \vec{z}_o)$ onto the current vectors $(\vec{x}_c, \vec{y}_c, \vec{z}_c)$ through the angle $\alpha$ through a rotation vector $\vec{a}$ (see figure 2.11.). The direction of $\vec{a}$ is the rotation axis and the magnitude is the tangent of half the rotation angle or $|\vec{a}| = \tan(\frac{\alpha}{2}) = \alpha$.

![Figure 2.11. Rotation Vector](image)

2.3.3. Net result of two rotations

The net result of a rotation defined by rotation vector $\vec{a}$ followed by a rotation defined by vector $\vec{b}$ does not observe the commutative law. The net result of a rotation defined by $\vec{a}$ followed by a rotation defined by $\vec{b}$ in that order is a third rotation

$$c = \frac{(\vec{a} + \vec{b} + \vec{b} \times \vec{a})}{(1 - \vec{a} \cdot \vec{b})}$$

(2.29.)

This derivation is given in appendix A. Equation (2.29.) is the so-called non-linear vector product discussed by Aharonov, Farach and Poole (1977). It is often thought that the order of two rotations is of little consequence. In other
words the net result of two rotations defined by \( \vec{a} \) and \( \vec{b} \) is often wrongly calculated to be the sum of the two vectors \( \vec{c} = \vec{a} + \vec{b} \), which is only true if \( \vec{a} \) and \( \vec{b} \) are small otherwise this equation represents an approximation. In the following example (figure 2.12.) a cylinder is initially placed in the \( x,y \)-plane, the axis of \( z \) being tangent to the cylinder’s centroidal axis. If \( \vec{a} \) is a rotation through \( \frac{\pi}{2} \) in the direction of the \( z \)-axis and \( \vec{b} \) is a rotation through \( \frac{\pi}{2} \) in the direction of the \( x \)-axis, the net result of \( \vec{a} \) followed by \( \vec{b} \) in this order is not the same as \( \vec{b} \) followed by \( \vec{a} \).

![Diagram showing rotations](image)

Figure 2.12. Net Result of 2 rotations

2.3.4. Rotation through a small angle
The net result of 2 rotations, the first one going through $\alpha(\vec{a})$ and the second one going through a small angle $\beta(\vec{b})$ so that $\tan(\frac{\beta}{2}) = \frac{\beta}{2}$, is derived in appendix A as:

$$
\vec{c} = a(1 + \frac{\vec{b}}{2}) + \frac{\vec{b}}{2} + \frac{\vec{b} \times a}{2}
$$

(2.30.)

2.3.5. Elements of the transformation matrix $[A]$

The rotation vector $\vec{a}(a_1,a_2,a_3)$ defines the orthogonal transformation matrix $[A]$ as shown in appendix A

$$
[A] = \cos \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1 - \cos \alpha}{a^2} \begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix} + \sin \alpha \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}
$$

(2.31.)

This result (2.31.) is also given by Beatty (1977). The elements of the orthogonal transformation matrix $A_{ij}$ are written as

$$
A_{ij} = \frac{(1 - a^2)\delta_{ij} + 2a_i a_j - 2\varepsilon_{ijk}a_k}{1 + a^2}
$$

(2.32.)

Where:

$\delta_{ij}$ Kronecker’s delta

$\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ if $i \neq j$

$\varepsilon_{ijk}$ 3D permutation pseudo tensor

$\varepsilon_{ijk} = 1$ if $i = 1, j = 2, k = 3$ or a permutation of these values

$\varepsilon_{ijk} = -1$ if $i = 3, j = 2, k = 1$ or a permutation of these values
\[ \varepsilon_{ij} = 0 \quad \text{if any two indices are equal} \]

The properties of transformation matrix \([A]\) are discussed in appendix A.

### 2.3.6. Rotation in one plane

The rotation of a vector \(\vec{x} (0,0,x_3)\) rotated through \(\alpha\) clockwise about the z-axis, gives the same results as the 2D rotation matrix for rotation in a plane (Mirsky (1955)). This is shown in appendix A:

\[
[A] = \cos \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{(1 - \cos \alpha)}{x_3^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_3^2 \end{bmatrix} + \frac{\sin \alpha}{x_3} \begin{bmatrix} 0 & -x_3 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(2.33.)

This shows that the proposed theorem of the rotation vector \(\vec{a}(a_1,a_2,a_3)\) is valid for rotations in one plane.

### 2.3.7. Rotation through \(\pi\)

Defining a rotation about an axis through an angle of near to \(\pi\), poses a specific problem as the rotational magnitude \(|\vec{a}| = \tan(\frac{\alpha}{2})\) has infinite value when \(\alpha = \pi\) (Buechter et al. (1992)). The elements of the rotation matrix \([A]\) cannot be derived as with the method explained earlier. The solution lies in defining the angle of rotation as \(\phi = \alpha - \pi\), the magnitude of the rotation vector \(f = \tan(\frac{\phi}{2})\) and the relationships \(\frac{f_i}{f} = \frac{a_i}{a}\). The transformation matrix elements \(A_{ij}\) are derived in appendix A:

\[
A_{ij} = \frac{(1 - f^2)\delta_{ij} + 2f_i f_k - 2\varepsilon_{kmn} f_m}{1 + f^2} [\delta_{ij} + \frac{2f_i f_j}{f^2}] \]

(2.34.)

### 2.3.8. Nodal rotations and axial displacement

Once the current member axis system is found by pre-multiplying the initial member axes system with matrix \([A]\), the angle of twist of the member \(\phi\) (in
figure 2.13.) and the basic rotational displacements of the element ends around the local member end vector \(x\) and \(y\) axes \(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}\) are generated by defining a vector \(\vec{p}\) from node 1 to node 2. (See appendix A).

\[
\phi = \frac{x_1 \cdot y_2 - x_2 \cdot y_1}{2}, \quad \theta_{x1} = \frac{y_1 \cdot \vec{p}}{L}, \quad \theta_{x2} = \frac{y_2 \cdot \vec{p}}{L}; \quad \theta_{y1} = -\frac{x_1 \cdot \vec{p}}{L}; \quad \theta_{y2} = -\frac{x_2 \cdot \vec{p}}{L};
\]

(2.35.)

Where:

\(L\) Deformed length of member

The total axial elongation \(e\) is the sum of the elongation due to the relative movement of the ends of the member \(e_a\) and the elongation due to bowing \(e_b\). It is shown in appendix A that

\[
e = \text{arclength} - L_0 = e_a + e_b
\]

\[
e = \frac{L^2 - L_0^2 + L_0^2}{2L_0} \left[ 4(\theta_{x1}^2 + \theta_{y1}^2) - 2(\theta_{x1} \theta_{x2} + \theta_{y1} \theta_{y2}) + 4(\theta_{x2}^2 + \theta_{y2}^2) \right]
\]
Note that this equation if found using only the first term of the binomial series and ignoring smaller terms of it. The member end forces and moments are found by differentiating the expression for the total strain energy of the member:

\[
U = \frac{EA}{2L_0} (e_a + e_b)^2 + \frac{EI_x}{2} \int \kappa_x^2 \, dl + \frac{EI_y}{2} \int \kappa_y^2 \, dl + \frac{GJ}{2L_0} \phi^2
\]

(2.37.)
Where:

- $G$ Modulus of Elasticity in shear
- $J$ Torsion Constant
- $L_0$ Original length of member
- $\kappa_x, \kappa_y$ Curvature around x-axis, y-axis
- $I_x, I_y$ Second Moment of Area
- $A$ Cross-sectional Area

Alternatively the non-linear effect due to the influence of axial force on the linear moment-curvature relations can be accounted for using stability functions, and the non-linear effect of bowing on the axial displacement modelled using correction factors (Ong (1992), Saafan (1963) and Wakefield (1980)).

### 2.3.9. Implementation into DR

The equilibrium of each node is sought in the DR process in terms of the balance between applied forces and moments on one hand and axial and shear forces and bending and torsional moments in the members on the other.

From the nodal rotations $\theta_{x1}, \theta_{y1}, \theta_{x2}, \theta_{y2}$ around the local member end vector axes, the member elongation $e$ and the angle of twist of the member $\phi$, the bending moments about the local end member $x$- and $y$-axis at the member ends $M_{x1}, M_{x2}, M_{y1}, M_{y2}$ and the torsional moment $M_\phi$ about the vector $\overrightarrow{p}$ can be found (Jennings (1977)). This is discussed more fully in appendix A. The force in the $x$, $y$ or $z$-direction resultant from these torsional and bending moments at node 1 and node 2 of the member, is found as:

$$\text{Resultant Force} = \frac{1}{L_0} \left( Fp_j + M_{x1}y_{1i} - M_{y1}x_{1i} + M_{x2}y_{2i} - M_{y2}x_{2i} \right)$$

(2.38.)

Note that this force is added to residual force components at node 1 of the link and subtracted from the residual force components at node 2. Similarly the resultant moments about the local $x, y$ or $z$ axis that needs to be added to the residual moment components at node 1 and node 2 are:
Resultant Moment applied to node 1 =

\[-\varepsilon_{ijk} \left( M_{x1} \frac{p_k y_{1j}}{L_0} - M_{y1} \frac{p_k x_{1j}}{L_0} + M_\phi \frac{(x_{1j} y_{2k} - y_{1j} x_{2k})}{2} \right)\]

Resultant Moment applied to node 2 =

\[-\varepsilon_{ijk} \left( M_{x2} \frac{p_k y_{2j}}{L_0} - M_{y2} \frac{p_k x_{2j}}{L_0} + M_\phi \frac{(x_{1j} y_{2k} - y_{1j} x_{2k})}{2} \right)\]

(2.39.)

All residuals of fixed, partially restrained rotational and partially restrained translational nodes are always set at zero.

The displacement stiffness of a node in the global \(x, y\) and \(z\) direction constitutes a 3x3 matrix assuming the node was not restrained in any direction but that the adjacent nodes were fixed. The displacement increments \([u]\) of a node are then directly related to the inverse of this stiffness matrix and to the nodal out-of-balance forces \([f]\) plus its previous displacement increments multiplied by a factor \(\leq 1.0\) to provide viscous damping. In appendix A the following expression is derived for the new displacement increments

\[u_k = C \frac{\varepsilon_{pq} \varepsilon_{ks} k_{pq} k_{qs} f_i}{2\Delta}\]

(2.40.)

Where:

\[\Delta = \frac{1}{6} \varepsilon_{pq} \varepsilon_{rs} k_{ij} k_{pq} k_{qs}\] Displacement Stiffness Matrix Determinant

\[C\] Viscous damping constant

\[k_{ij}\] Element of stiffness matrix

At this stage, the kinetic energy in the DR process can be found and if it is maximum, the previous displacement increments are not carried over.
2.4. Conclusion

A review of the development and formulation of the Dynamic Relaxation process was outlined as an appropriate method for the form-finding and load analysis of non-linear structures. The basis of the method consists of tracing the motion of each node of the structure step by step for small time increments until (due to artificial damping) the structure comes to static equilibrium. As the analysis of spline supported membrane structures certainly includes geometric and material non-linearities (e.g. membrane on/off slackening and spline buckling), a combined kinetic and viscous damping approach was chosen for this thesis. The viscous damping will damp out the overall behaviour while imposed high residual forces will be treated by the kinetic damping. The different element types that will be used in the applications in chapter 5 were briefly discussed. A summary was given of the six degrees of freedom analysis that incorporates beam elements in the DR process. The method is based on the concept of describing any rotation (or sequence of rotations) as a simple rotation about a rotation vector. In Chapter 4 this method will be used, together with analytical expressions, to validate the original three degrees of freedom spline theory presented in Chapter 3, through a series of test cases.