Optimal Battery Capacity

Andrés

April 12, 2011

Abstract

A simple model to find the optimal battery capacity to maximize flight time is presented. To estimate flight times, some parameters of the airframe are necessary, but can for a first iteration be estimated roughly for typical airframes. The choice of battery capacity is astonishingly simple, as it turns out to be independent of any airframe parameter beside the ratio of airframe weight to battery weight. The optimal choice is when the batteries weight twice as much as the airframe without batteries. An analysis of the best compromise between the reductions in flight time and total weight is also presented. The best compromise is for batteries that weight about 83% of the airframe. For this mass fraction, about 87% of the most optimal flight time is still achieved and this with only 61% of the total weight. A minimal bound for the battery weight is also determined. Batteries that weight less than 20% of the airframe, produce a larger penalty in flight time than the improvement in total weight. The model is shown to be valid also for helicopters, as only the efficiency factor change and not the optimality conditions. The report brings then two typical examples of fixed-wing aircraft and ends giving some words of caution regarding the choice of battery capacity.

1 Mathematical model

1.1 Fixed-wing aircraft

1.1.1 Flight time

In level flight the weight of the airplane and the thrust have to be equal to the lift and drag, respectively

$$m\,g = \frac{1}{2}\,\rho\,v^2\,C_L\,A\ , \tag{1}$$

$$T = \frac{1}{2}\,\rho\,v^2\,C_D\,A\ , \tag{2}$$

where m is the mass, $g = 9.81\,m\,s^{-2}$ the acceleration constant, $\rho = 1.225\,kg\,m^{-3}$ the air density, v the velocity of flight, C_L the lift coefficient, A the wing area, T the thrust and C_D the drag coefficient. Using some algebraic conversions, we find relations for the needed thrust and the resulting flight velocity

$$T = m\,g\,\frac{C_D}{C_L}\ , \tag{3}$$

$$v = \sqrt{\frac{2\,m\,g}{\rho\,C_L\,A}}\ . \tag{4}$$

Taking into consideration that, the power needed to keep level flight is $P = T\,v$, the above equations can be combined to render

$$P = m^{\frac{3}{2}} \sqrt{\frac{2\,g^3}{\rho\,A}\,\frac{C_D}{C_L^{\frac{3}{2}}}}\ . \tag{5}$$

The energy needed to fly a certain time is given by

$$E(t) = \int_0^t \frac{P}{\eta}\,d\tau \approx \frac{P}{\eta}\,t\ , \tag{6}$$

where η is the total propulsion efficiency. In the last step, efficiency and needed power were assumed to be time independent. This equation delivers a maximum flight time t_{max} for a given total amount of electric energy carried E_{max}

$$t_{max} = \frac{E_{max}\,\eta}{P}\ . \tag{7}$$

Using Eq. (5), we find

$$t_{max} = \frac{E_{max}\,\eta}{m^{\frac{3}{2}}\sqrt{\frac{2\,g^3}{\rho\,A}\,\frac{C_D}{C_L^{\frac{3}{2}}}}}\ . \tag{8}$$

However, the mass m of the airplane is related to the energy carried. We assume the following simple linear relation

$$m = m_0 + \alpha\,E_{max}\ , \tag{9}$$

where m_0 is the mass without batteries and α is the reciprocal of the specific energy density of the batteries. Inserting this equation delivers finally a relation between flight time and battery energy

$$t_{max} = \frac{E_{max}}{(m_0 + \alpha\,E_{max})^{\frac{3}{2}}}\,\eta\,\sqrt{\frac{\rho\,A}{2\,g^3}\,\frac{C_L^{\frac{3}{2}}}{C_D}}\ . \tag{10}$$

Flight time is proportional to the propulsion efficiency η and to the sink rate efficiency $C_L^{\frac{3}{2}}/C_D$ of the airframe. If the design goal is to achieve a long flight time, then airframes with good sink rate efficiencies $C_L^{\frac{3}{2}}/C_D$ should be used. It is in general important to reach high propulsion efficiencies η and to minimize the battery independent weight m_0. Flight times can also be increased via a reduction of wing loading m/A, which means in essence that the wing area is increased for a given total mass.

The factor in Eq. (10) containing E_{max} is the one that determines which capacity will be optimal. The other factors represent more or less efficiencies and determine the absolute time of flight that will be achieved. These factors are difficult to determine, as they depend on non-trivial properties, such as the aerodynamics of the airframe. However, when it comes to determining the optimal capacity, only the mass ratio between empty weight and battery weight is needed.

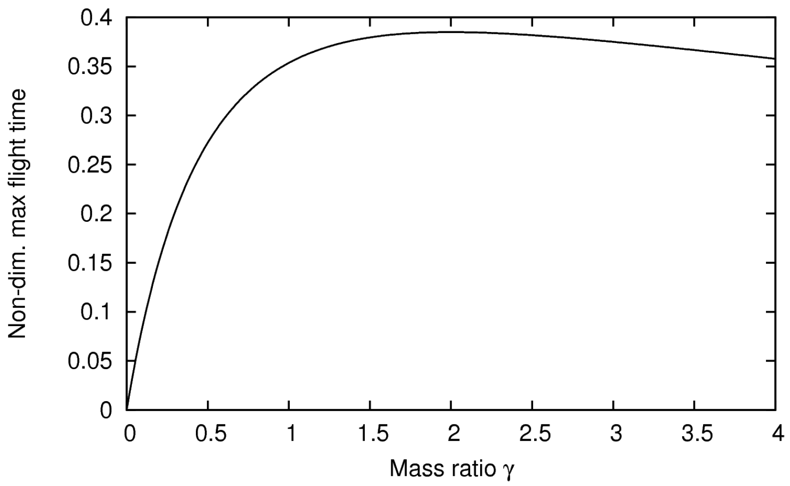

Figure 1: Non-dimensionalized flight time $\hat{t}_{max}$ in dependence of mass ratio γ. Optimal mass ratio is $\gamma = 2$, i.e. the batteries have double the weight of the empty airframe.

Taking Eq. (10), introducing the mass ratio $\gamma = \alpha\, E_{max}/m_0$ and doing some algebraic transformations, one obtains

$$t_{max} = \frac{\gamma}{(1+\gamma)^{\frac{3}{2}}} \, \frac{\eta}{\alpha} \sqrt{\frac{\rho\, A}{2\, m_0\, g^3} \, \frac{C_L^{\frac{3}{2}}}{C_D}} \; . \tag{11}$$

The time is non-dimensionalized to be able to emphasize which capacity is optimal

$$\hat{t}_{max} = t_{max} \frac{\alpha}{\eta} \sqrt{\frac{2\, m_0\, g^3}{\rho\, A} \, \frac{C_D}{C_L^{\frac{3}{2}}}} \; .$$

Eq. (11) simplifies into

$$\hat{t}_{max} = \frac{\gamma}{(1+\gamma)^{\frac{3}{2}}} \; , \tag{12}$$

which depends only on the mass ratio γ. Maximization in relation to γ is achieved by determining the roots of the first derivative and in principle also by double-checking the sign of the second derivative. We will stick only to the first step $\frac{d}{d\gamma}\hat{t}_{max} = 0$. First we determine the derivative

$$\frac{d}{d\gamma}\hat{t}_{max} = \frac{\gamma}{(1+\gamma)^{\frac{3}{2}}} = \frac{1}{(1+\gamma)^{\frac{3}{2}}} - \frac{3}{2} \, \frac{\gamma}{(1+\gamma)^{\frac{5}{2}}} \; , \tag{13}$$

and then we calculate the root

$$(1+\gamma) - \frac{3}{2}\gamma = 0 \; ,$$

and find finally

$$\gamma = 2 \; . \tag{14}$$

The results has an astonishing simplicity. It means that the battery should be chosen to have twice the empty weight m_0 of the airframe (having everything besides the batteries installed). By doing this, one will automatically have the optimal capacity for longest flight times. Of course, to optimize even more the flight time, the airframe should be trimmed to maximize $C_L^{\frac{3}{2}}/C_D$. Fig. 1 presents a plot of Eq. (12).

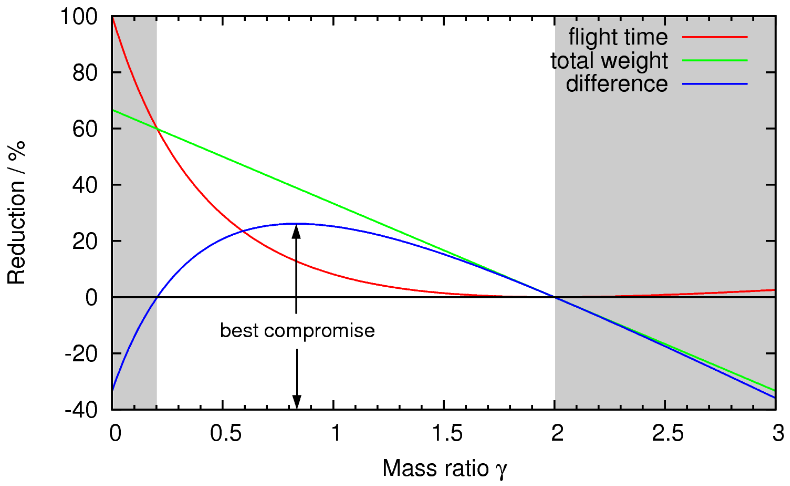

Figure 2: Relative changes in flight time, total airframe weight and the difference between the reductions. The shaded areas mark the mass ratio choices to be avoided.

The function (12) is very flat around the optimal mass ratio $\gamma = 2$. The non-dimensionalized time $\hat{t}_{max}$ assumes there the value

$$\hat{t}_{max}\big|_{\gamma=2} = \frac{2}{\sqrt{27}} = 0.3849\ldots$$

This value represents the longest flight time possible. Changing the battery capacity and, hence, the mass ratio, will decrease the flight time. Due to the flat nature of this maximum, it might not make much sense to use $\gamma = 2$, as the battery costs increase over-proportionally to the benefit in flight time. On the one hand, decreasing battery mass (=capacity) has a positive effect on stall speed and climb rates, while on the other, it decreases flight time. The total weight of the airframe can be expressed by γ and m_0

$$m = (1 + \gamma)\, m_0 \ . \tag{15}$$

Hence, the total weight of the airframe is $3\, m_0$ at the optimum mass ratio $\gamma = 2$. The relative change in total airframe weight in relation to the optimal mass, is given by

$$\frac{\Delta m}{3\, m_0} = \frac{3\, m_0 - (1 + \gamma)\, m_0}{3\, m_0} = 1 - \frac{1}{3}(1 + \gamma) \ . \tag{16}$$

For simplicity, the difference was chosen to be positive for a decrease in mass. The relative change in flight time is given by

$$\frac{\Delta \hat{t}_{max}}{\hat{t}_{max}\big|_{\gamma=2}} = 1 - \frac{\sqrt{27}}{2} \frac{\gamma}{(1+\gamma)^{\frac{3}{2}}} \ . \tag{17}$$

Here again, the difference was set to be positive for a decrease in flight time.

Fig. 2 presents a plot of the relative changes in flight time, total weight and their difference. This figure shows that there is a mass ratio, for which the reductions in flight time and total weight are most apart (i.e. largest difference). The value at which this happens is $\gamma \approx 0.833$. For this mass ratio choice, the penalty in flight time is small compared to the reduction in total weight and is, thus, a reasonable compromise between flight time and total airframe weight. The airframe achieves 87% of the best flight time with only 61% of the total weight. Fig. 2 shows also the choices of mass ratios γ, which do not make much sense. Going further than $\gamma = 2$ reduces the flight time and should be considered only if a larger range is more important. The figure shows also that for $\gamma < 0.2$ the penalty in flight time is larger than the improvement in total weight, representing a very suboptimal choice. Note that the proposed battery capacity/weight for small RC-Airplanes is often dangerously near to this value.

1.1.2 Range

The calculation to obtain the optimal capacity for range is similar to the above case. The main difference is that there is not really an optimal capacity, as the range becomes always larger for an increasing battery capacity. The range is given for a constant velocity by $r = v\,t$, which means that

$$r_{max} = \frac{E_{max}\,\eta\,v}{P} = \frac{E_{max}\,\eta}{T} = \frac{E_{max}\,\eta}{m\,g}\,\frac{C_L}{C_D} \ , \tag{18}$$

and hence

$$r_{max} = \frac{E_{max}}{m_0 + \alpha\,E_{max}}\,\frac{\eta}{g}\,\frac{C_L}{C_D} \ . \tag{19}$$

The factor containing E_{max} achieves its maximum at $E_{max} \to \infty$, which means that increasing battery capacity will result always in a higher range. In a true situation, a design will try to attain a large range without having to much penalties in flight time, climb rates, stall speeds, etc. (see comments below in Sec. 3).

1.2 Helicopters

Helicopters produce both lift and thrust by using one or more rotors. Therefore, a different equation system than (1) is needed. For simplicity we restrict our considerations to the simpler case of hovering, in which the thrust of the rotor is used completely to equilibrate gravity. Using Glauert's hypothesis, the thrust produced by the rotor is

$$T = 2\,\rho\,A\,\overline{V}\,w \ , \tag{20}$$

where A is the area of the disc area, w is the induced velocity and $\overline{V}$ is the resultant velocity

$$\overline{V} = \sqrt{(w - V\,\sin\alpha)^2 + (V\,\cos\alpha)^2} \ , \tag{21}$$

with α equal to the angle-of-attack of the rotor plane. Now, considering only the hovering case, $V = 0$, so that $\overline{V} = w$, and hence

$$T = 2\,\rho\,A\,w^2 \ . \tag{22}$$

For steady hovering gravity and thrust equilibrate

$$m\,g = 2\,\rho\,A\,w^2 \ , \tag{23}$$

which delivers an expression for the induced velocity

$$w = \sqrt{\frac{m\,g}{2\,\rho\,A}} \ . \tag{24}$$

The power needed is given by

$$P = D\,V + T\,w \ , \tag{25}$$

where D is the drag. However, considering $V = 0$ and $T = m\,g$, this Eq. renders

$$P = T\,w = \sqrt{\frac{m^3\,g^3}{2\,\rho\,A}} \ . \tag{26}$$

Eq. (7) is still valid and using the expression for the needed power, we obtain

$$t_{max} = \frac{E_{max}}{m^{\frac{3}{2}}}\,\eta\sqrt{\frac{2\,\rho\,A}{g^3}} \ . \tag{27}$$

The battery mass is again assumed to be linearly dependent on E_{max} [see Eq. (9)], which delivers

$$t_{max} = \frac{E_{max}}{(m_0 + \alpha\,E_{max})^{\frac{3}{2}}}\,\eta\sqrt{\frac{2\,\rho\,A}{g^3}} \ . \tag{28}$$

The factor containing E_{max} is the same as in the case of a fixed wing. Therefore, the analysis presented before is still valid and the non-dimensionalized time $\hat{t}_{max}$ is the same as given in Eq. (12). Hence the optimal mass ratio is still $\gamma = 2$. For the sake of completeness, we write down the expression relating t_{max} and the mass ratio

$$t_{max} = \frac{\gamma}{(1+\gamma)^{\frac{3}{2}}}\,\frac{\eta}{\alpha}\,\sqrt{\frac{2\,\rho\,A}{m_0\,g^3}} \ . \tag{29}$$

2 Examples

Two typical examples are presented here. The first one is a larger civilian UAV, where the area is about $1\,m^2$ which equals a span of ca. $3\ m$. The empty mass m_0 of such an airframe is about 5 kg. The lift coefficient for best sink rate is about $C_L \approx 1$, while an estimate for the drag is $C_D \approx 0.1$, so that $C_L^{\frac{3}{2}}/C_D \approx 10$. The energy density of Lithium-ion polymer batteries is 130 to 200 $W\,h\,kg^{-1}$ rendering $\alpha \approx 6 \cdot 10^{-3}\ kg\,W^{-1}\,h^{-1}$. The propulsion efficiency can be estimated to be $\eta = 0.4$, by considering all losses from the batteries to the

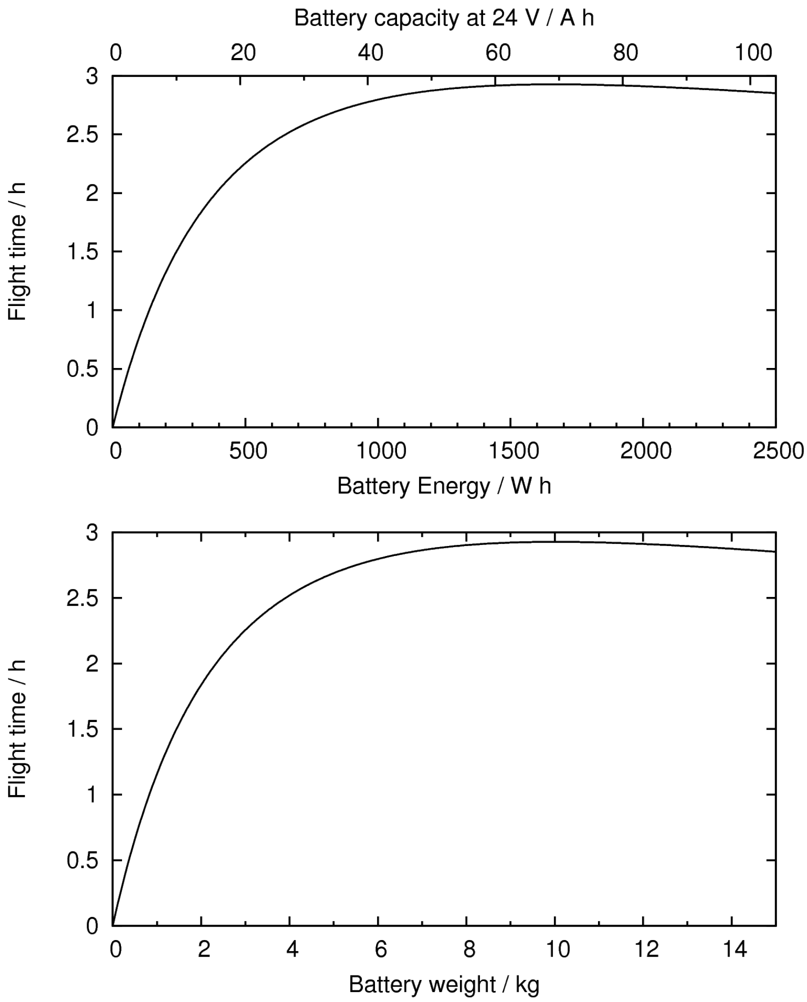

Figure 3: Top: Flight time vs. battery energy and battery capacity. Bottom: Flight time vs. battery weight.

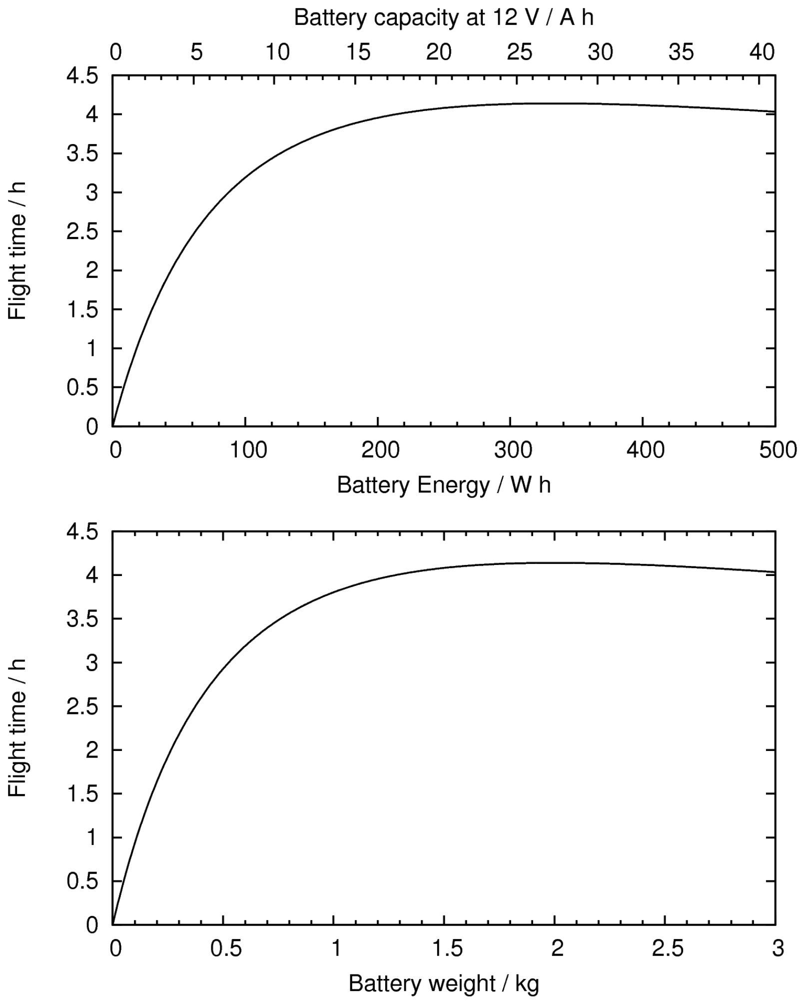

Figure 4: Optimality for *EasyGlider* (Mutliplex Modellsport GmbH & Co. KG, Bretten, Germany) type of airframe. Top: Flight time vs. battery energy and battery capacity. Bottom: Flight time vs. battery weight.

motor. Plugging this into Eq. (10) and plotting t_{max} against E_{max} delivers the graphs found in Fig. 3.

The region around the optimal energy/capacity is very shallow, so that a large increase in capacity (cost) has only a small effect on flight time. For the above example, going much beyond $20A\,h$ does not make sense, as costs increase over-proportional to the benefit obtained.

Consider now a smaller airplane, such as an *EasyGlider* (Mutliplex Modellsport GmbH & Co. KG, Bretten, Germany). Mostly the battery independent mass m_0 and the wing area have to be adapted. We assume here $m_0 = 1\ kg$ and $A = 0.4\ m^2$. The results for such an airframe are presented in Fig. 4. Remember here that the usual battery for an *EasyGlider* is a three cell Lithium-ion polymer battery with a capacity of about $2500\ mA\,h = 2,5\ A\,h$. Therefore, the usual capacity is still far away from the optimal value for long flight times.

3 Some words of caution

The flight times calculated above depend on efficiency parameters, such as η and $\frac{C_L^{\frac{3}{2}}}{C_D}$, which can vary from airframe to airframe substantially. So, do not take the times too seriously. However, the model shows that, the optimum energy/capacity depends only on the ratio between empty mass and battery mass: $\gamma = m_{bat}/m_0 = 2$. Therefore, although you wont be able to make accurate predictions on the actual flight time, you can easily choose the battery capacity to optimize flight time.

Also to mention is that, other parameters besides flight time are important in deciding battery capacity. The motor/battery combination has to be selected to be able to manage the expected take off weight. An airplane needs relative small power to keep level flight, but climbing can be quiet costly. If you increase battery capacity, and hence weight, without considering take-off and climbing situations, your airframe might have overweight and the motor might not have enough power to do necessary maneuvers. Also, consider that the stall speed rises for increasing wing loading.

The above used flight condition is the one that is optimal for flight time in the sense that the sink rate is minimal. This condition is very near to stall, so that caution is needed not to stall the airframe during maneuvers. In general, the airframe will be trimmed somewhat faster to have some security margin.