Recent Progress in Dynamical Systems Theory

Ralph Abraham*

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Abstract

In *The Foundations of Mechanics*, Chapter Five, we presented a concise summary of modern dynamical systems theory from 1958 to 1966. In this short note, we update that account with some recent developments.

Introduction

There are by now a number of histories of modern dynamical systems theory, typically beginning with Poincaré. But there are few accounts by the participants themselves. Some exceptions are, in chronological order:

- Steve Smale, *The Mathematics of Time*, 1980
- David Ruelle, *Change and Chaos*, 1991
- Ed Lorenz, *The Essence of Chaos*, 1993

And also, there are autobiographical remarks by Steve Smale, Yoshi Ueda, Ed Lorenz, Christian Mira, Floris Takens, T. Y. Li, James Yorke, and Otto Rossler, and myself in *The Chaos Avant-garde*, edited by Yoshi Ueda and myself (2000).

In this brief note I am adjoining to this historical thread my own account of the impact, around 1968, of the experimental findings (the Ueda, Lorenz, and Mira attractors of 1961, 1963, and 1965, resp.) on the global analysis group around Steve Smale and friends. I am grateful to the Calcutta Mathematical Society, especially to Professors H. P. Majumdar and M. Adhikary, for extraordinary hospitality and kindness extended over a period of years during my several stays in Kolkata.

*Mathematics Department, University of California, Santa Cruz, CA 95064. *rha@ucsc.edu*

What the global analysis group did accomplish before its demise in 1968 was brought together and extended in the "three authors book", *Invariant Manifolds* of Hirsch, Pugh, and Shub, in 1977.

This note expands somewhat on my earlier remarks. For example, I wrote,

> Structural stability was an earlier candidate for a generic property of vectorfields. Although it turned out to be generic only in the two-dimensional case, it may be of considerable importance in the applications ... Some weaker notion of stability may be found to be generic eventually. [1]

This was written in 1966, at the same time that I published the results of my years-long study of the generic properties of vectorfields (Abraham, 1967b). In the global analysis group, we all knew that structural stability was not a generic property in dimensions above two, but the crisis of chaotic attractors had not yet struck. This came in July 1968, at the AMS Summer School on Global Analysis in Berkeley.[2]

The following summer, there was another important gathering of the clan at the University of Warwick. Remarking further there on yhe fundamental problem that I christened *the yin-yang problem*, I wrote,

> Last year at the Berkeley Summer Institute on Global Analysis there were 20 talks on differential equations, of which 10 were concerned with the "yin-yang problem". Some large (yin) sets of differential equations with generic properties are known, some small (yang) sets which can be classified are known, but in general the two domains have not yet met. They began to approach each other, but recently progress degenerated into a sequence of conjectures and counter-examples without further approach. For guidance on future progress, the *I Ching* was consulted on the question: Will the current program lead to a solution of the yin-yang problem ... The prophesy obtained was hexaxgram 13 (Breakthrough) changing to 49 (Revolution), interpreted as *No.*
>
> This year at the Warwick Summer School there were 22 talks of which 3 were concerned with the yin-yang problem. The *I Ching* was again consulted (using for the first time, half-crowns). As it had become clear through the esoteric Buddhist principles of Karma and Transcendence that the yes-no question formerly posed was too restrictive, the question

[1](Abraham, 1967; p. 174)
[2](Chern and Smale, 1970)

> asked this time was: How will the subject evolve in the course of the next year? ...
>
> The answer to the question was given by the *I Ching* as hexagram 10 (Conduct) transforming to 11 (Peace), a clear indication of the future dominance of yin. ... Although the yin and yang aspects of differential equations will continue to oppose each other as prophesied last year, a harmonious balance between the conflicting forces may be obtained by proper study.[3]

This prediction that slow progress would lead to improvement came true after three decades or so, in the notion of singular hyperbolicity, in developments that I will outline in this note. I am going to begin with the prehistory of global anaysis, the influence of Solomon Lefschetz, as this is sometimes overlooked.

By the way, a more detailed history may be found in my contribution to *The Chass Avant-garde*[4] and here I will correct a few inaccuracies.

The Prehistory: 1940-1958

In 1954 I went to Ann Arbor, Michigan, to get an undergraduate degree in electrical engineering. Needing a job to cover expenses, I was lucky to find work in the laboratory of Don Glaser, a young professor of physics who was building the first large bubble chamber. During the years 1954-1958 I was the engineer on this project, and this led me to become a graduate student in physics. In 1958 the project was finished. Glaser moved to Berkeley and into biophysics, while I moved into the mathematics Ph.D. program. In 1960 Don was awarded a Nobel Prize for his inventions.

Math came as a natural progression, and because I had become a student and teaching assistant of Nate Coburn, who was a technical master of the difficult art of tensor geometry, and a demanding and superb teacher. At that time, the University of Michigan was home to some great mathematicians, such as Norman Steenrod and Raoul Bott. It was in a class with Bott that I first learned of the new index-free movement in differential geometry and differential topology. The translation of differential geometry and relativity theory into the new notation system became part of my Ph.D. thesis work with Coburn, and in the summer of 1959 I went to Mexico City to write up my thesis in a novel environment. I went to the Universidad Nacional Autonoma de Mexico (UNAM) to ask permission to sit daily in the math library, and that is where I first met Lefschetz, then about 75 years old.

[3] (Abraham, 1969; p. 163)

[4] (Abraham and Ueda, 2000; p. 81)

I knew already of his pioneering work in algebraic topology, his influential topology texts of 1924 and 1942, and his career at Princeton University and at the *Annals of Mathematics*. It was my impression that he was a liberal and a philanthropist, and that he wished to spread the mathematical frontier into Latin America by creating a beachhead in Mexico. Beginning during the Second World War, he would spend half of each year in Mexico City, and by 1959 he had a flourishing Ph.D. program well underway at UNAM. While there I noticed there was a conference on dynamical systems theory taking place, and saw some of the principals, although I would not meet them until a year or so later. Steve Smale recounts his first meeting in Mexico City in 1956, after finishing his thesis with Bott at the University of Michigan.[5] It was under Lefschetz' influence that Peixoto began his work on structural stability in dimension two, which became the spark for the golden days of global analysis at Berkeley, beginning in the Fall of 1960. Smale and Thom were also involved in Peixoto's program on structural stability. It was again my good luck to arrive in the math department in Berkeley, for my first academic job, just as Smale, Hirsch, Thom, Chern, and many other mathematicians of international repute, came together there.

In the larger picture, the tradition of Poincaré in dynamical systems theory bifurcated into two streams, one in Russia with Liapunov, Andronov, and so on, the other in the USA with Birkhoff. The latter came to a premature end while the Russian school thrived. The special role of Lefschetz was to transplant the Russian thread back into the USA. This was another great philanthropy of his, and it began around 1942 with his reading of the Russian literature. Although French was his first language (he grew up in Paris and moved to the USA at age 21 where he did his graduate work in math) he became fluent in Russian, and eventually translated the key Russian texts into English.

Here is a chronology of the events in Lefschetz' work relating to dynamical systems theory.

- 1942, Lefschetz began reading ODE
- 1943, Lefschetz translated Kryloff and Bogoliuboff
- 1944, Lefschetz begins program in Mexico City
- 1946, Lefschetz published his book, *Lectures on ODE*
- 1949, Lefschetz translated Andronov and Chaikin
- 1953, Lefschetz retired from Princeton University
- 1956, Dynamics conference in Mexico City (with Smale, Thom, and Hirsch)
- 1958, Lefschetz joined the Research Institute for Advanced Study (RIAS)

[5] See Smale, in: (Abraham and Ueda, 2000; p. 2)

- 1958, Lefschetz influenced Peixoto, Peixoto met Smale
- 1959, Peixoto's theorem published in Lefschetz' journal
- 1959, Dynamics conference in Mexico City (Lefschetz met Smale)
- 1959, Smale to IMPA (Peixoto's institute in Rio)

Well, all this leads up to the special role of the Russian concept of structural stability in the history of modern dynamical systems theory, to which we now turn.

The First Ten Years: 1958-1968

So the new subject was well under way in the fall of 1960, when I arrived in Berkeley, and the golden age of global analysis began. After working alone in my office in Campbell Hall for a few weeks, I found out that there was a daily tea upstairs. I went up, and immedately met Steve Smale and Moe Hirsch. Steve and I found we had Michigan in common, and he invited me to his office to discuss his current work, which was on handlebody theory. His work on the horseshoe map was also under way, involving the homoclinic points of Poincaré. Soon we began a joint project on homoclinic tangles, studying the early works of Poincaré, and Birkhoff and Smith. This project culminated in a series of papers by Steve on stable manifolds, and on diffeomorphisms with infinitely many periodic points. Our collaboration continued until 1968, ending with a counterexample in four dimensions that we presented in the infamous summer of 68.

In retrospect, some of our work was off target and is forgettable. Among the main accomplishments, in my personal view, are these.

- 1961, Smale, stable manifold theorem (announced in Urbino)
- 1961, myself, transversality theorem (announced in Bonn)
- 1962, my lectures on transversality theory (at Columbia)
- 1964, Thom's catastrophe theory began to appear
- 1966, my lectures at Princeton on mechanics and on transversality were published
- 1967, Smale survey paper in the *Bulletin of the AMS*
- 1968, counterexample to genericity conjecture (joint with Smale)
- 1968, Berkeley Summer School, crisis of chaotic attractors
- 1971, Ruelle and Takens, application to turbulence
- 1977, Hirsch, Pugh, and Shub, *Invariant Manifolds* published

The central idea of the global analysis program for dynamical systems theory, mainly due to Smale and set out in his review paper of 1967, was this (stated here in the version for diffeomorphisms). The *nonwandering set* (which is normally the closure of the set of periodic points, where all the action is) should generically be a finite union of basic sets which are hyperbolic, and thus subject to the all-powerful stable manifold theorem. Also, the stable and unstable manifolds of these basic sets should intersect transversally, and thus the global dynamics would be structurally stable, that is, unchanged by small perturbations.

All this seems reasonable enough, for diffeomorphisms. However, for vectorfields, as we discovered when the Lorenz attractor finally came to our attention around 1966, further complications may be introduced by the presence of rest points (that is, zeros, aka singularities) in the nonwandering set. This is the essence of our crisis of 1968, when many of us scurried off to study various applications and experimental results of ODEs.

The Lorenz Attractor Problem

Unfortunately, as we were totally immersed in the ambience of differential topology and global analysis from 1960 through 1968, we completely missed the emergence on the scene of the chaotic attractors of Ueda (1961), Lorenz (1963), and Mira (1965). The Ueda attractor was discovered during an analog computer study of a forced oscillator, easily reduces to a cascade of diffeomorphisms of the plane, and is amenable to study by the global analysis methods of the 1960s. The Mira attractor is a semi-cascade or iteration of a noninvertible map of the plane, and is out of the range of these methods.[6]

So it was the Lorenz attractor of 1963 that frustrated the global analysis school in 1968. It is a vectorfield that does not easily reduce to a diffeomorphism, for the attractor includes a rest point at the origin. It resisted proof of the properties considered essential for a chaotic attractor. And worst of all, it did not seem to be structurally stable, although it seemed to be robust in some sense. The Lorenz equations define a family of vectorfields with three parameters, σ, r, and b. The usual values are $10, 8/3$, and 28, respectively. When σ and b are held at these values and r is increased from 0 to above 313, a sequence of apparently stable bifurcations is displayed.[7]

Around 1982 I received a small package in the mail from H. Bruce Stewart of

[6]But see (Abraham, Gardini, and Mira, 1997) for some methods that do apply.

[7]These are exhaustively studied in (Sparrow, 1982).

$$
\begin{aligned}
x' &= -\sigma x + \sigma y \\
y' &= -xz + rx - y \\
z' &= xy - bz
\end{aligned}
$$

the Applied Math Department of the Brookhaven National Laboratory. Within I found a reel of 16 mm film, one of the greatest examples of computer-graphic visual mathematics of all time. It showed the bifurcation sequence of the Lorenz system with large portions of the insets and outsets of the rest points and periodic orbits as r increases from zero to large values. The onset of chaos leading up to $r = 28$ was clearly shown. Under the influence of this film, Chris Shaw and I created the explanation of the case $r = 28$ (in eight pages and eight four-color hand drawings) in our visual text in the 1980s.[8] One of these drawings is shown in Figure 1.[9] Subsequently, Bruce Stewart presented a few of his computer graphics in monochrome, in his book with J. M. T. Thompson.[10]

Early attempts to understand the structure of the Lorenz attractor by John Guckenheimer and Bob Williams[11] nearly succeeded, and eventually were vindicated. Abstracting some geometric properties from the structure of the Lorenz attractor, Guckenheimer defined an archetype, the *geometric Lorenz attractor*, which was amenable to the methods of global analysis.

Here is the essence of the geometric model. In Figure 2[12] we have a solid triangular block. We imagine a flow downward through the block, and coming outwards from each of the two triangular faces. Now imagine the two triangular faces flowing outward, then up and around, and down into T again, entering through S, the square top of T. This is indicated in Figure 3.[13] An excellent description of this geometry may be found in (Hirsch, Smale, and Devanay, 2004; Sec. 14.3).

The problem then became to show that the Lorenz attractor was a geometric Lorenz attractor, and this was listed as an outstanding problem (the 14th) by Steve Smale in 1998. The solution to this outstanding problem was provided by Warwick

[8] (Abraham and Shaw, 1992; pp. 383-390)

[9] (Abraham and Shaw, 1992; p. 384)

[10] (Thompson and Stewart, 1986; pp. 227-234)

[11] (Guckenheimer, 1976), (Guckenheimer and Williams, 1979), (Williams, 1979)

[12] Figure 1 from (Guckenheimer and Williams, 1979)

[13] Figure 2 from (Guckenheimer and Williams, 1979)

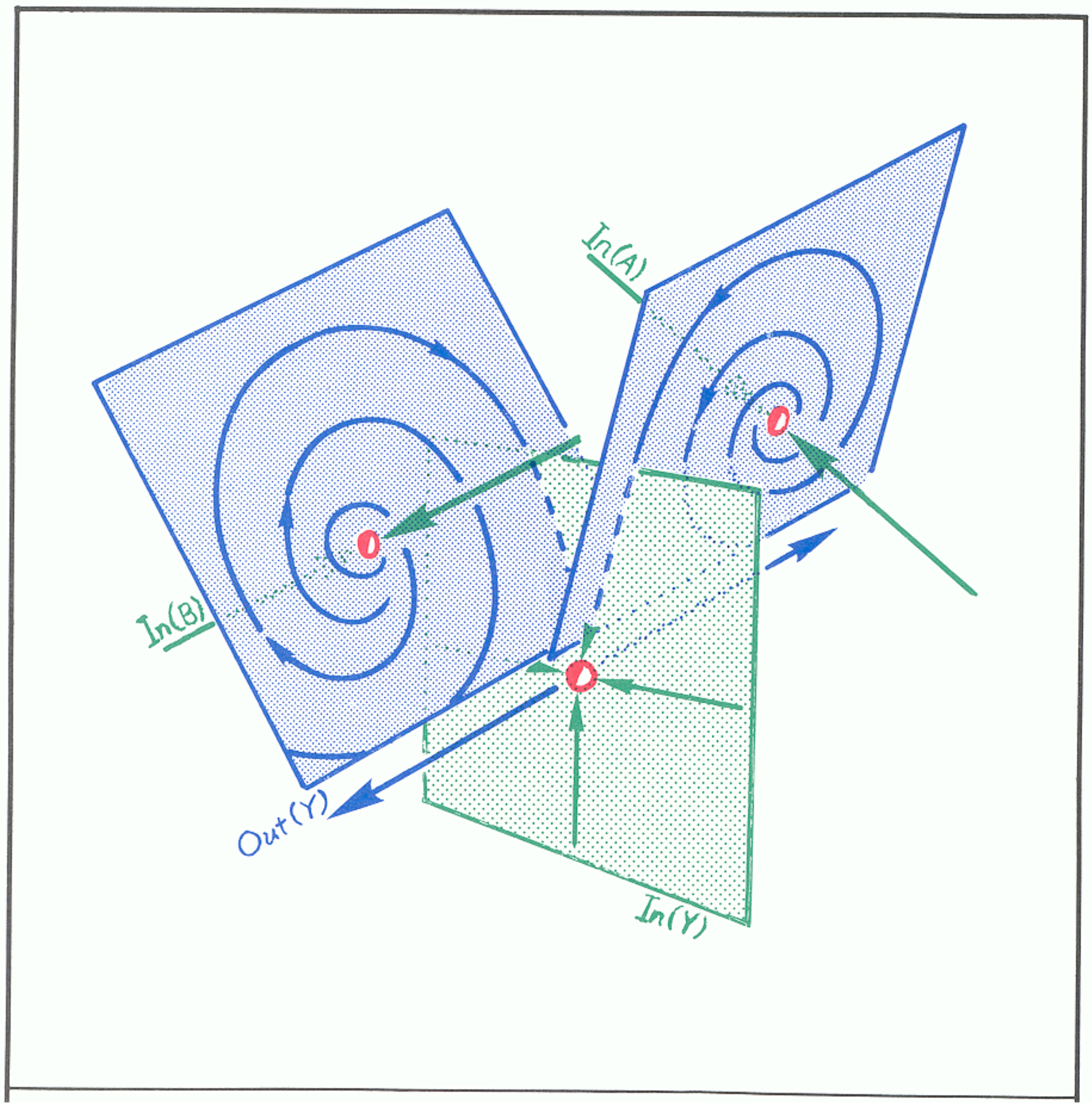

Figure 1: Outsets of the Lorenz Attractor.

Tucker.[14]

The Lorenz Attractor Solution

Jacob Palis came from Rio to Berkeley as a graduate student, finished his Ph.D. with Steve Smale in 1967, and thus was subject to the great hopes of the golden age, and also the frustrations of the Lorenz problem. He became an international figure in his own right, and succeeded Peixoto as Director of IMPA in Rio. He made many important contributions to dynamical systems theory in the global analysis tradition, including joint works with most of the main figures, including Smale, Hirsch, Pugh, Shub, Newhouse, Robinson, Sullivan, and Takens. His personal influence, and the ambience of IMPA, have figured importantly in the solution of the Lorenz problem. The main points of his overall program, growing over a period of years, was eventually published.[15] Among his Ph.D. students are several senior figures of dynamical systems theory, for example, de Melo and Mañé, and more recently, important participants in the solution of the Lorenz problem, such as Morales, Pacifico, Pujals, and Viana. It is significant that Warwick Tucker, the Australian who solved the Lorenz problem in his Ph.D. thesis at Uppsala University in 1998, under Carleson, was a postdoctoral fellow at IMPA, 1998-2000.

In his detailed history of the Tucker solution, Marcelo Viana says,[16]

> . . . a new theory has been emerging on how such a global theory could be developed. In this direction, a comprehensive program was proposed a few years ago by J. Palis, built on the following core conjecture: *every smooth dynamical system (diffeomorphism of flow) on a compact manifold can be approximated by another that has only finitely many attractors, either periodic or strange.*[17]

A crucial step in this direction was taken by Morales, Pacifico, and Pujals, who introduced a weakening of the hyperbolicity idea of Smale's stable manifold theorem, called *singular hyperbolicity*. This pertains particularly to three-dimensional flows with rest points (singularities).[18] And they proved that a robust attractor containing a rest point is singular hyperbolic, and has the properties of a geometric Lorenz attractor.

[14] See (Tucker, 1999), (Viana, 2000), and (Stewart, 2000).
[15] (Palis, 2000)
[16] (Viana, 2000; p. 9)
[17] (Palis. 2000)
[18] (Morales, Pacifico, and Pujals, 1998)

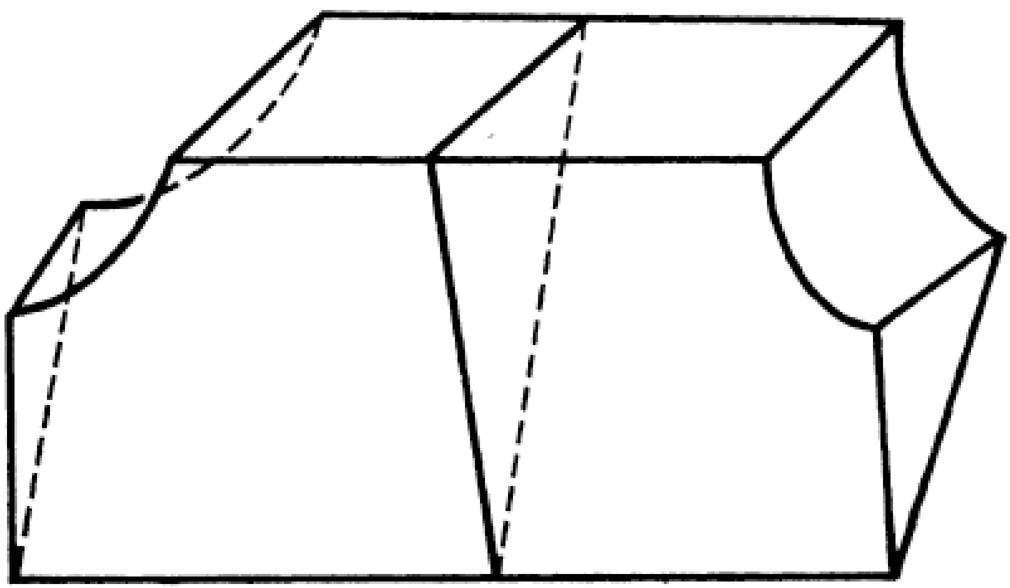

FIG. I

Figure 2: The basic 3-cell, T.

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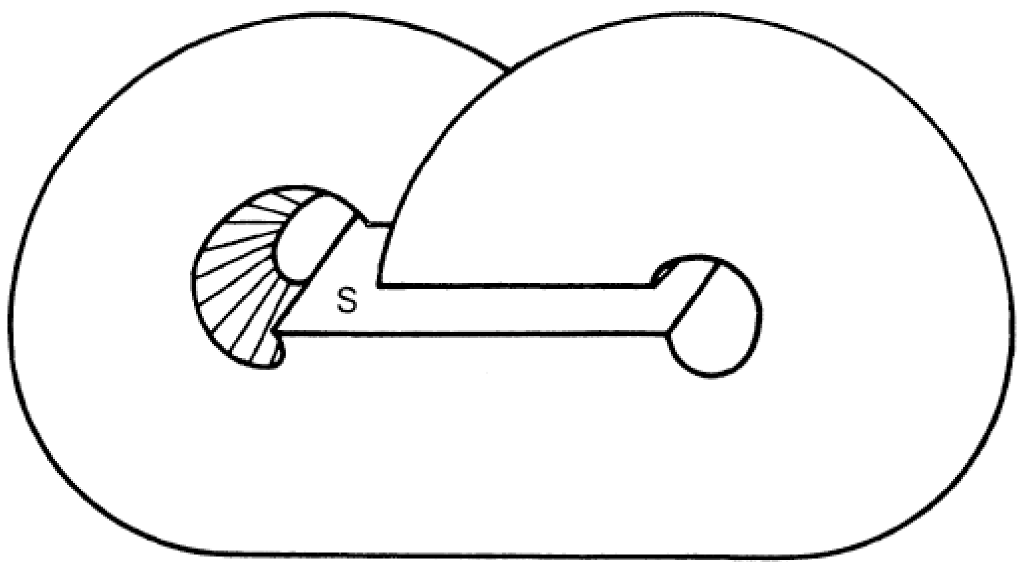

FIG. 2

Figure 3: The flow reenters T through its top square, S.

Thus, it remained only to show that the Lorenz attractor actually is a robust strange attractor. And that is what Tucker accomplished in his thesis of 1998, making use of ultra-sophisticated computational methods, and bringing the crisis of global analysis to an end.

Conclusion

After this short story of the golden decade of modern dynamical systems theory (1958-1968), the crisis of 1968, the dark ages of three decades (1968-1998), and the spectacular resolution of Tucker in 1998, built upon the work of Palis and his students at IMPA.

It is important to remark that the crisis, and also the solution, depended upon sophisticated computational work.[19] The computer revolution has merged with the differential topology tradition of Poincaré,

The prediction of the *I Ching* has come true!

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[19] See also (Aubin and Dalmedico, 2002).

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