

Mth101 SUBJECTIVE COMPOSED BY SADIA ALI SADI www.vustudent.ning

Question No: 24 (Marks: 10)

Evaluate the following limit Q # 1 Whether the given lines are parallel, perpendicular or none of these?

$$\frac{1}{2}(y-1) = x-3 \text{ and } 8-2y = x+7$$

Solution:

$$L_1: \frac{1}{2}(y-1) = x-3$$

$$L_2: 8-2y = x+7$$

First we have to calculate the slope.

$$L_1: \frac{1}{2}(y-1) = x-3$$

$$y-1 = 2(x-3)$$

$$y = 2x - 6 + 1$$

$$y = 2x - 5$$

Comparing it with equation of line $y = m x + c$

$$\text{Slope of } L_1 = m_1 = 2$$

$$L_2: 8 - 2y = x + 7$$

$$-2y = x + 7 - 8$$

$$-2y = x - 1$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Comparing it with equation of line $y = m x + c$

$$\text{Slope of } L_2 = m_2 = -1/2$$

Hence the given two lines are perpendicular because $m_1 m_2 = (2)$

$$\left(-\frac{1}{2}\right) = -1$$

Q # 2

Let

$$f(x) = \frac{x}{x+3} \quad \text{and} \quad g(x) = x^2$$

Find whether $(f \circ g)(x)$ and $(g \circ f)(x)$ are equal or not?

Solution:

Here

$$f(x) = \frac{x}{x+3} \quad \text{and} \quad g(x) = x^2$$

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g)(x) = f(x^2)$$

$$(f \circ g)(x) = \frac{x^2}{x^2 + 3}$$

Also

$$(g \circ f)(x) = g(f(x))$$

$$(g \circ f)(x) = g\left(\frac{x}{x+3}\right)$$

$$(g \circ f)(x) = \frac{x^2}{(x+3)^2}$$

$$(g \circ f)(x) = \frac{x^2}{x^2 + 6x + 9}$$

Hence $(f \circ g)(x) \neq (g \circ f)(x)$

Q # 3 Determine whether the equation represents a circle, if the equation represents a circle, find the center and radius?

$$x^2 + 2x + y^2 - 4y = 5$$

Solution:

First, group the x-terms, group the y-terms, and take the

constant to the right side:

$$(x^2 + 2x) + (y^2 - 4y) = 5$$

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 5 + 1 + 4$$

Then write the left side as squares and add up the right side and you get

$$(x+1)^2 + (y-2)^2 = 10,$$

and now you can find the center and radius and graph it. Here the center would be $(-1, 2)$ and the radius would be $\sqrt{10}$. So to graph it, all

you need to do is find the point $(-1, 2)$ and then plot the points $\sqrt{10}$ up,

$\sqrt{10}$ down, $\sqrt{10}$ to the right, and $\sqrt{10}$ to the left of it, and draw the circle

through them.

Q # 4 Solve the inequality

$$-6 \leq \frac{4-2x}{3} < 1$$

Solution:

$$-6 \leq \frac{4-2x}{3} < 1$$

multiply by 3, we get

$$-18 \leq 4 - 2x < 3$$

subtracting by 4

$$-22 \leq -2x < -1$$

divide by -2

$$11 \geq x > \frac{1}{2}$$

$$\text{or } \frac{1}{2} < x \leq 11$$

Hence the required solution is

$$\left(\frac{1}{2}, 11 \right]$$

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$$\lim_{y \rightarrow -2} g(y) \text{ where, } g(y) = \begin{cases} y^2 + 5 & \text{if } y < -2 \\ 3 - 3y & \text{if } y \geq -2 \end{cases}$$

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Question No.1

Use implicit differentiation to find dy/dx if

$$x^2 = \frac{x+y}{x-y}$$

Solution:

$$x^2 = \frac{x+y}{x-y}$$

$$\frac{d(x^2)}{dx} = \frac{(x-y)\frac{d(x+y)}{dx} - (x+y)\frac{d(x-y)}{dx}}{(x-y)^2}$$

$$2x = \frac{(x-y)(1 + \frac{dy}{dx}) - (x+y)(1 - \frac{dy}{dx})}{(x-y)^2}$$

$$2x = \frac{x-y + x\frac{dy}{dx} - y\frac{dy}{dx} - x + y + x\frac{dy}{dx} + y\frac{dy}{dx}}{(x-y)^2}$$

$$2x = \frac{-2y + 2x\frac{dy}{dx}}{(x-y)^2}$$

$$2x(x-y)^2 = -2y + 2x\frac{dy}{dx}$$

$$x(x-y)^2 = -y + x\frac{dy}{dx}$$

$$x(x-y)^2 + y = x\frac{dy}{dx}$$

$$\frac{x(x-y)^2 + y}{x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x(x-y)^2 + y}{x}$$

Some other method can be used to solve this.

Question No. 2.

Find the slope of the tangent line to the given curve at the specified point

$$2(x^2 + y^2)^2 = 25(x^2 - y^2); \quad (3,1)$$

Solution:

$$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$

differentiating with respect to 'x'

$$2 \times 2(x^2 + y^2) \frac{d(x^2 + y^2)}{dx} = 25 \frac{d(x^2 - y^2)}{dx}$$

$$4(x^2 + y^2)(2x + 2y \frac{dy}{dx}) = 25(2x - 2y \frac{dy}{dx})$$

$$4(x^2 + y^2) \cdot 2(x + y \frac{dy}{dx}) = 2(25x - 25y \frac{dy}{dx})$$

$$8(x^2 + y^2)(x + y \frac{dy}{dx}) = 2(25x - 25y \frac{dy}{dx})$$

$$4(x^2 + y^2)(x + y \frac{dy}{dx}) = (25x - 25y \frac{dy}{dx})$$

$$4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = 25x - 25y \frac{dy}{dx}$$

$$4y(x^2 + y^2) \frac{dy}{dx} + 25y \frac{dy}{dx} = -4x(x^2 + y^2) + 25x$$

$$4y(x^2 + y^2) \frac{dy}{dx} + 25y \frac{dy}{dx} = -x(4x^2 + 4y^2) + 25x$$

$$y(4x^2 + 4y^2) \frac{dy}{dx} + 25y \frac{dy}{dx} = x(-4x^2 - 4y^2 + 25)$$

$$y(4x^2 + 4y^2 + 25) \frac{dy}{dx} = x(-4x^2 - 4y^2 + 25)$$

$$\frac{dy}{dx} = \frac{x(-4x^2 - 4y^2 + 25)}{y(4x^2 + 4y^2 + 25)}$$

At point (3,1)

$$\frac{dy}{dx}\Big|_{(3,1)} = \frac{x(-4x^2 - 4y^2 + 25)}{y(4x^2 + 4y^2 + 25)}$$

$$\frac{dy}{dx}\Big|_{(3,1)} = \frac{3(-4 \times 3^2 - 4 \times 1^2 + 25)}{1(4 \times 3^2 + 4 \times 1^2 + 25)}$$

$$= \frac{3(-36 - 4 + 25)}{36 + 4 + 25}$$

$$\frac{dy}{dx}\Big|_{(3,1)} = \frac{-3 \times 15}{40 + 25}$$

$$= \frac{-45}{65}$$

$$= \frac{-9}{13}$$

Question No. 3

A 10-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at the rate of 2-ft/sec, how fast will the foot be moving away from the wall when the top is 6 ft above the ground?

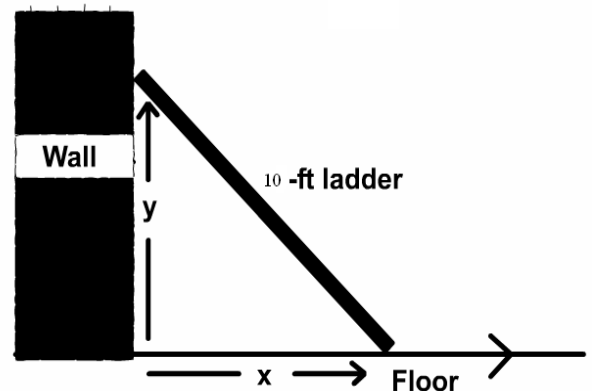
Solution:

Let

x = Distance in feet between wall and foot of the ladder

y = Distance in feet between floor and top of the ladder

t = number of seconds after the ladder starts to slip.



$$\frac{dy}{dt} = -2 \text{ ft/sec}$$

It is negative because y is decreasing as time increases, since the ladder is slipping down the wall.

$$\frac{dx}{dt} = ?$$

Using Pythagoras Theorem,

$$x^2 + y^2 = 10^2$$

differentiating w.r.t 't'

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

when $y = 6 \text{ ft}$

$$\frac{dx}{dt} = -\frac{6}{x} \frac{dy}{dt} \dots\dots\dots(i)$$

When the top is 6 ft above the ground

$$x^2 + 6^2 = 10^2$$

$$x^2 = 100 - 36$$

$$x^2 = 64$$

$$x = 8$$

put in (i) we get

$$\frac{dx}{dt} = -\frac{6}{8} \frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{6}{8} (-2)$$

$$= \frac{12}{8}$$

$$= \frac{3}{2} \text{ ft/sec}$$

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Question No.1 :

Find the relative extreme values of the function

$$f(x) = a \sin x + b \cos x$$

Solution:

Since,

$$f(x) = a \sin x + b \cos x$$

$$f'(x) = a \cos x - b \sin x \dots\dots\dots(i)$$

$$\text{put } f'(x) = 0 \Rightarrow a \cos x - b \sin x = 0$$

$$\Rightarrow a \cos x = b \sin x$$

$$\Rightarrow \frac{a}{b} = \tan x$$

$$\Rightarrow \tan x = \frac{a}{b} \Rightarrow x = \tan^{-1}\left(\frac{a}{b}\right)$$

By Pythagorus theorem

$$\Rightarrow \sin x = \frac{a}{\pm\sqrt{a^2+b^2}} \quad , \quad \cos x = \frac{b}{\pm\sqrt{a^2+b^2}}$$

Now, taking second derivative

$$f''(x) = -a \sin x - b \cos x \dots\dots\dots(ii)$$

$$\text{put } \sin x = \frac{a}{\sqrt{a^2+b^2}} \quad , \quad \cos x = \frac{b}{\sqrt{a^2+b^2}} \text{ in (ii)}$$

$$f''(x) = -a \frac{a}{\sqrt{a^2+b^2}} - b \frac{b}{\sqrt{a^2+b^2}}$$

$$f''(x) = -\left(\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}\right) = -\sqrt{a^2+b^2} < 0$$

$$\text{So } f(x) \text{ has relative maxima at } \sin x = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos x = \frac{b}{\sqrt{a^2+b^2}}$$

and since both sin and cos are **positive in first quadrant** so we say f has relative maxima at

$$x = \frac{\pi}{2} - \tan^{-1}\left(\frac{a}{b}\right)$$

$$\text{put } \sin x = \frac{-a}{\sqrt{a^2+b^2}} \quad , \quad \cos x = \frac{-b}{\sqrt{a^2+b^2}} \text{ in (ii)}$$

$$f''(x) = a \frac{a}{\sqrt{a^2+b^2}} + b \frac{b}{\sqrt{a^2+b^2}}$$

$$f''(x) = \left(\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}\right) = \sqrt{a^2+b^2} > 0$$

$$\text{So } f(x) \text{ has relative minima at } \sin x = \frac{-a}{\sqrt{a^2+b^2}} \text{ and } \cos x = \frac{-b}{\sqrt{a^2+b^2}}$$

and since both sin and cos are **positive in third quadrant** so we say f has relative minima at

$$x = \frac{3\pi}{2} - \tan^{-1}\left(\frac{a}{b}\right)$$

Question No.2:

Let

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$$

Does Mean Value Theorem hold for f on $\left[\frac{1}{2}, 2\right]$?

Solution:

To check whether f satisfies M.V.T

I. Clearly f is continuous on $\left[\frac{1}{2}, 2\right]$.

II. To check f is differentiable on $\left[\frac{1}{2}, 2\right]$

We check whether $f'(1)$ exists.

$$L.H.Limt f'(1) = \lim_{x \rightarrow 1-0} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1-0} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \rightarrow 1-0} x + 1 = 2$$

$$R.H.Limt f'(1) = \lim_{x \rightarrow 1+0} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1+0} \frac{x - 1}{x - 1} = 1$$

Thus $L.H.Limt f'(1) \neq R.H.Limt f'(1)$.

Therefore, $f'(1)$ does not exist and the M.V.T. does not hold on $\left[\frac{1}{2}, 2\right]$.

Question No.3 :

Integrate the following

$$(i) \frac{xa^{x^2}}{x^2} \quad (ii) \frac{\ln x}{x}$$

Solution:

$$(i) \int \frac{xa^{x^2}}{x^2} dx$$

$$\text{put } x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}$$

$$\int \frac{a^t dt}{2t} = \frac{1}{2} \int \frac{a^t}{t} dt = \frac{1}{2} \int \left[\frac{1}{t} (1 + t \ln a + \frac{t^2}{2!} (\ln a)^2 + \frac{t^3}{3!} (\ln a)^3 + \dots) \right]$$

$$(\because a^x = 1 + x \ln a + \frac{x^2}{2!} (\ln a)^2 + \frac{x^3}{3!} (\ln a)^3 + \dots)$$

$$= \frac{1}{2} \int \left[\frac{1}{t} + \ln a + \frac{t}{2!} (\ln a)^2 + \frac{t^2}{3!} (\ln a)^3 + \dots \right]$$

$$= \frac{1}{2} (\ln t + t \ln a + \frac{t^2}{4} (\ln a)^2 + \frac{t^3}{18} (\ln a)^3 + \dots)$$

$$= \frac{1}{2} (\ln x^2 + x^2 \ln a + \frac{(x^2)^2}{4} (\ln a)^2 + \frac{(x^2)^3}{18} (\ln a)^3 + \dots)$$

$$(ii) \int \frac{\ln x}{x} dx$$

$$\text{put } \ln x = t \Rightarrow \frac{1}{x} dx = dt$$

$$\int t dt = \frac{t^2}{2} + c$$

By back substitution

$$\int t dt = \frac{(\ln x)^2}{2} + c$$

Question 1

Marks 8

Find area of region enclosed by given curves

$$y = 3x - 4x^2 + x^3, \quad y = 0, x = 0, x = 3.$$

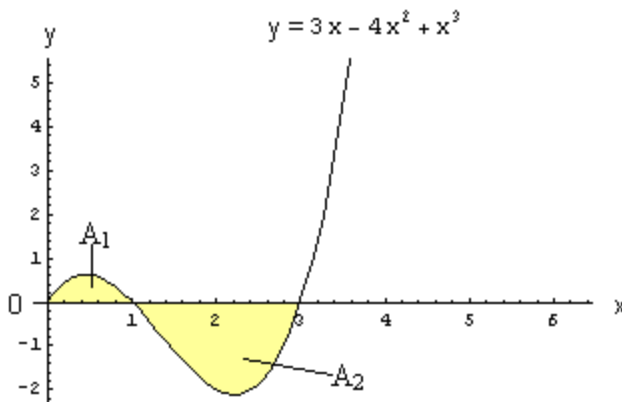
Solution:

$$y = x^3 - 4x^2 + 3x, \quad y = 0, x = 0, x = 3.$$

$y = 0$ is the x-axis, $x = 0$ is y-axis and $x = 3$ is a line parallel to y-axis crossing x-axis at point 3.

So we have to find out the area bounded by the curve $y = x^3 - 4x^2 + 3x$ and the x-axis over the interval $[0, 3]$

Graph of $y = 3x - 4x^2 + x^3$ is below



We have to find the area of shaded region. It is clear from the graph that total area, A , under the curve in interval $[0, 3]$ is divided into two regions A_1 and A_2 . In $[0, 1]$, A_1 is

above x-axis and in $[1, 3]$, A_2 is below x-axis. So

$$A = A_1 - A_2 = \int_0^1 (x^3 - 4x^2 + 3x) dx - \int_1^3 (x^3 - 4x^2 + 3x) dx$$

NOTE: Since $A_2 = \int_1^3 (x^3 - 4x^2 + 3x) dx$ is below x-axis, so we will get negative value of integral. That's why we subtract A_2 from A_1 .

Even if we don't have the graph, we can come to this conclusion by first finding the zero of a function that lie between given interval $[0,3]$.

Zero of a function is a value of x which makes a function f(x) equal to zero.

$$x^3 - 4x^2 + 3x = 0$$

$$x(x^2 - 4x + 3) = 0$$

$$x(x^2 - 3x - x + 3) = 0$$

$$x\{x(x-3) - 1(x-3)\} = 0$$

$$x(x-3)(x-1) = 0$$

$$x = 0, 1, 3$$

So zero of a function $(x^3 - 4x^2 + 3x)$ is 0, 1 and 3

Now 1 lie between our given interval $[0, 3]$, which means that total area is divided into two regions. One in interval $[0,1]$ and other in $[1,3]$

So

$$\begin{aligned}
A &= A_1 - A_2 = \int_0^1 (x^3 - 4x^2 + 3x) dx - \int_1^3 (x^3 - 4x^2 + 3x) dx \\
&= \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right) \Big|_0^1 - \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right) \Big|_1^3 \\
&= \frac{1^4}{4} - \frac{4(1)^3}{3} + \frac{3(1)^2}{2} - \left(\frac{0^4}{4} - \frac{4(0)^3}{3} + \frac{3(0)^2}{2} \right) - \\
&\quad \left[\frac{3^4}{4} - \frac{4(3)^3}{3} + \frac{3(3)^2}{2} - \left(\frac{1^4}{4} - \frac{4(1)^3}{3} + \frac{3(1)^2}{2} \right) \right] \\
&= \frac{1}{4} - \frac{4}{3} + \frac{3}{2} - \left[\frac{81}{4} - \frac{108}{3} + \frac{27}{2} - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) \right] \\
&= \frac{5}{12} + \frac{9}{4} + \frac{5}{12} \\
&= \frac{37}{12}
\end{aligned}$$

Question 2

Evaluate

Marks 7

$$\int_0^{\infty} x e^{-x^2} dx$$

Solution:

$$\int_0^{\infty} x e^{-x^2} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \int_0^t (-2x) e^{-x^2} dx$$

$$= -\frac{1}{2} \lim_{t \rightarrow \infty} \left| e^{-x^2} \right|_0^t$$

$$= -\frac{1}{2} [0 - 1] = \frac{1}{2}$$

Question 3

Marks 10

Evaluate

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$$

Solution:

Consider,

$$\frac{1}{(x^2 + a^2)(x^2 + b^2)} = \frac{Ax + B}{x^2 + a^2} + \frac{Cx + D}{x^2 + b^2} \dots \dots \dots (1)$$

$$\Rightarrow 1 = (Ax + B)(x^2 + b^2) + (Cx + D)(x^2 + a^2)$$

$$\Rightarrow 1 = Ax^3 + Bx^2 + Axb^2 + Bb^2 + Cx^3 + Dx^2 + Cxa^2 + Da^2$$

$$\Rightarrow 1 = (A + C)x^3 + (B + D)x^2 + (Ab^2 + Ca^2)x + (Bb^2 + Da^2)$$

Comparing coefficients

$$A + C = 0 \quad , \quad B + D = 0$$

$$Ab^2 + Ca^2 = 0 \quad , \quad Bb^2 + Da^2 = 1$$

Here, $C = -A \Rightarrow Ab^2 - Aa^2 = 0$

$$\Rightarrow A(b^2 - a^2) = 0 \Rightarrow A = 0 \quad \text{if} \quad b^2 - a^2 \neq 0$$

$$\Rightarrow C = 0$$

Since, $B + D = 0 \Rightarrow D = -B$

thus $\Rightarrow Bb^2 - Ba^2 = 1 \Rightarrow B(b^2 - a^2) = 1 \Rightarrow B = \frac{1}{(b^2 - a^2)}$

$$\Rightarrow D = \frac{-1}{(b^2 - a^2)}$$

So (1) becomes

$$\begin{aligned} \frac{1}{(x^2 + a^2)(x^2 + b^2)} &= \frac{(0)x + 1/(b^2 - a^2)}{x^2 + a^2} + \frac{(0)x - 1/(b^2 - a^2)}{x^2 + b^2} \\ &= \frac{1}{(b^2 - a^2)(x^2 + a^2)} - \frac{1}{(b^2 - a^2)(x^2 + b^2)} \\ &= \frac{1}{b^2 - a^2} \left[\frac{1}{x^2 + a^2} - \frac{1}{x^2 + b^2} \right] \end{aligned}$$

integrating both sides,

$$\begin{aligned} \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} &= \frac{1}{a^2 - b^2} \left[\int_0^\infty \frac{dx}{(x^2 + b^2)} - \int_0^\infty \frac{dx}{(x^2 + a^2)} \right] \\ &= \frac{1}{a^2 - b^2} \lim_{t \rightarrow \infty} \left[\int_0^t \frac{dx}{(x^2 + b^2)} - \int_0^t \frac{dx}{(x^2 + a^2)} \right] \\ &= \frac{1}{a^2 - b^2} \lim_{t \rightarrow \infty} \left[\frac{1}{b} \tan^{-1}\left(\frac{x}{b}\right) - \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]_0^t \\ &= \frac{1}{a^2 - b^2} \lim_{t \rightarrow \infty} \left[\frac{1}{b} \tan^{-1}\left(\frac{t}{b}\right) - \frac{1}{a} \tan^{-1}\left(\frac{t}{a}\right) \right] \\ &= \frac{1}{a^2 - b^2} \lim_{t \rightarrow \infty} \left[\frac{\pi}{2b} - \frac{\pi}{2a} \right] = \frac{\pi}{2ab(a + b)} \end{aligned}$$

Question No.1

Marks 10

Find the volume of the solid that results when the region enclosed by the given curve is revolved about the x-axis

$$y = \sqrt{25 - x^2}, \quad y = 3$$

Solution:

We know the volume formula, when we revolve $y = f(x)$ about x-axis, is given by

$$\int_a^b \pi([f(x)]^2 - [g(x)]^2) dx$$

Here given that $y = \sqrt{25 - x^2}$ and $y = 3$.

Now to find the initial and final limits of integrations we shall solve both of curves simultaneously to find their points of intersection. On equating both functions we have

$$\begin{aligned}\sqrt{25 - x^2} &= 3 \\ \Rightarrow 25 - x^2 &= 9 \\ \Rightarrow x^2 &= 16 \\ \Rightarrow x &= \pm 4 \\ \Rightarrow x &= -4, 4\end{aligned}$$

Hence given $a = -4$, and $b = 4$, so we have

$$\begin{aligned}\int_a^b \pi([f(x)]^2 - [g(x)]^2) dx &= \int_{-4}^4 \pi([\sqrt{25 - x^2}]^2 - [3]^2) dx \\ &= 2\pi \int_0^4 (25 - x^2 - 9) dx = 2\pi \int_0^4 (16 - x^2) dx \\ &= 2\pi \left[16x - \frac{x^3}{3} \right]_0^4 \\ &= 2\pi [64 - 64/3] \\ &= \frac{256\pi}{3}\end{aligned}$$

Question No. 2**Marks 8**

Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

$$y = \sqrt{x}, x = 4, x = 9, y = 0$$

Solution:

To find the volume when we revolve around y-axis, we integrate with respect to x i.e.

$$V = \int_a^b 2\pi x f(x) dx$$

the curve is $y = \sqrt{x}$ so

$$V = \int_a^b 2\pi x f(x) dx = \int_a^b 2\pi x [\sqrt{x}] dx$$

Here $a = 4$, and $b = 9$

Thus

$$\begin{aligned} V &= \int_4^9 2\pi x [\sqrt{x}] dx \\ &= 2\pi \left[\int_4^9 x^{3/2} dx \right] \\ &= 844\pi / 5 \end{aligned}$$

Question No.3

Find the area of the surface generated by revolving the given curve about the y -axis

$$x = \sqrt{16 - y}, 0 \leq y \leq 15$$

Solution:

Here given that

$$\begin{aligned}
x &= g(y) = \sqrt{16-y}, & 0 \leq y \leq 15 \\
\Rightarrow g'(y) &= -\frac{1}{2\sqrt{16-y}} \\
\Rightarrow (g'(y))^2 &= \frac{1}{4(16-y)} \\
\Rightarrow 1+(g'(y))^2 &= 1+\frac{1}{4(16-y)} \\
\Rightarrow &= \frac{65-4y}{4(16-y)} \\
\Rightarrow \sqrt{1+(g'(y))^2} &= \sqrt{\frac{65-4y}{4(16-y)}}
\end{aligned}$$

Since

$$S = \int_0^{15} 2\pi x \sqrt{1+(g'(y))^2} dy$$

So, by above values

$$\begin{aligned}
S &= 2\pi \int_0^{15} (\sqrt{16-y}) \left(\sqrt{\frac{65-4y}{4(16-y)}} \right) dy \\
&= (65\sqrt{65} - 5\sqrt{5}) \frac{\pi}{6}
\end{aligned}$$

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